What to Expect when Everyone is Expecting: Self-Fulfilling Expectations and Asset-Pricing Puzzles

Nicolae Gârleanu
UC Berkeley-Haas and NBER

Stavros Panageas
UCLA, Anderson School of Management and NBER*

January 2020

Abstract

We study an economy without bubbles in which expectations about future discount rates can become self-fulfilling because asset valuations redistribute wealth across different investor cohorts. For such redistribution to take place, the wealth of arriving and existing cohorts must react differently to discount rates, and in addition only the existing agents be marginal in financial markets. The self-fulfilling nature of discount rate expectations means that the economy can address several well documented empirical asset pricing facts (excessive volatility, return predictability, low interest rate level and volatility), while all real quantities (aggregate consumption and dividend growth) are smooth.

Keywords: asset pricing, self-fulfilling expectations, sunspot equilibria, equity premium puzzle, excess volatility puzzle, inequality
JEL Classification: G01, G12

*We are grateful for comments and suggestions from seminar and conference participants at Boston College, Chicago Booth, Chicago FED, Kellogg, Minneapolis FED, NBER, Notre Dame, Princeton, UT Austin, and Yale SOM.

1 Introduction

We study an economy in which expectations about future discount rates can become self-fulfilling because asset valuations redistribute wealth between different cohorts. For such redistribution to take place, the endowments of arriving and existing cohorts of investors must react differently to discount rates, and in addition only the existing agents be marginal in financial markets. The self-fulfilling nature of discount rate expectations means that the economy can address several well documented empirical asset pricing facts (excessive volatility, return predictability, low interest rate level and volatility), while all real quantities (aggregate consumption and dividend growth) are locally deterministic processes.

The possibility of self-fulfilling expectations (also referred to as "indeterminacy") in overlapping generations (OLG) endowment economies, such as the one we study, has long been recognized. The indeterminacy in these models, however, stems from the existence of a (rational) bubble. We contribute to this literature by showing that indeterminacy can arise in the absence of bubbles, and regardless of the conditions the existence of bubbles requires.[1] The relation between a model with bubbles and ours is that both models assume multiple sources of wealth to allow discount rate movements to have redistributive effects. We show, however, that there is no need for one of these sources of wealth to be a bubble.

The ability to dispense with bubbles imparts an additional element of discipline and testability. Without bubbles, conventional present value relations (and accordingly Campbell-Shiller decompositions) continue to hold. Thus, whereas the shocks driving the volatility of asset prices may be unobservable, the relation between discount rates, dividend growth, and prices is the same as in any other asset-pricing model. The model can therefore be confronted with data, while being agnostic about the origin of the shocks.

We next provide a more detailed outline of the model and summarize its main implications. The framework is a continuous-time OLG economy. Agents arrive (and die) continuously. Upon birth, they are endowed with "human capital," which we model as an income stream that they receive throughout their lives. In addition, some agents have the ability to create a new firm, which partially displaces the output of some existing firms. In creating the new firms, they face a dilemma.

[1] Abel et al. (1989) shows (in the production version of such economies) that for bubble to exist the transfers from the household sector to the corporate sector for the purposes of investment must exceed the transfers from the corporate to the household sector in the form of profits (or, more appropriately, return on capital). In the data, however, the latter transfers have consistently been larger.

They could choose to start a "safe" firm, which is bound to produce positive dividends, or take a gamble and start a "risky" firm. A risky firm is a lottery. If successful, it produces higher dividends than the safe firm in perpetuity. If unsuccessful, it fails at inception, and the entrepreneur has to rely on her human capital to finance her consumption. Importantly, there are no "real" shocks to the economy, i.e., no exogenous shocks to technologies or preferences.

Restricting attention to deterministic equilibria, we show that in this rather minimal model the price of a firm can be indeterminate. Specifically, when new-firm creation responds sufficiently to discount rates, there are multiple (indeed, a continuum of) equilibrium paths that can take the economy to its steady state. The logic centers on the interaction between discount-rate anticipations and wealth redistribution. Suppose, for instance, that all investors anticipate low discount rates in the future. These low discount rates benefit arriving cohorts disproportionately, by raising the attractiveness of the risky-firm choice and thus spurring the creation of new firms, which displace a larger fraction of the dividend of the old firms. Accordingly, the wealth share of the owners of the older firms (i.e, the older cohorts) declines and so does their consumption growth. The reduction in the consumption growth of older cohorts (who are marginal in financial markets) leads to a lower equilibrium discount rate, confirming the agents' anticipation.

To summarize, there are two key elements in the above argument: a) discount rates redistribute wealth between two groups of agents; and b) discount rates are determined by the consumption growth of one group (existing cohorts), but not the other (arriving cohort).

To better explain this point, we revisit the indeterminacy that obtains in models with bubbles and show that a similar redistribution mechanism is responsible for this indeterminacy. We argue that the existence of a bubble is not necessary for indeterminacy in the sense that a bubble is useful only insofar as it provides a second asset that causes redistribution between the different cohorts. The fact that this second asset is fundamentally worthless is actually immaterial.

Because of the multiplicity of deterministic equilibria, it is fairly straightforward to construct stochastic, "sunspot" equilibria. In such equilibria the source of randomness pertains to which equilibrium agents coordinate on. In all these equilibria asset prices can have essentially arbitrary volatility, while the consumption processes for all investors are "locally deterministic" (i.e., exhibit no instantaneous volatility). Even though the change in an agent's consumption over the next increment of time is known, the "long run" change in the agent's consumption is stochastic. Hence, as long as investors have recursive preferences, the extrinsic uncertainty — rooted in the random

shifts of expectations — carries a risk premium.

While the primary goal of the paper is to illustrate the qualitative feedback effects between redistribution and discount rate expectations that arise in models with heterogeneous agents, we also provide a quantitative illustration of the model. The goal of this exercise is to show that, while the shocks are non-fundamental and not directly observed, the rest of the model structure provides enough discipline that the model can be confronted with its empirical predictions, just like any other asset pricing model. We show that the model produces realistic risk premiums, return predictability patterns, low interest rate levels and volatility, even though consumption and dividends are locally deterministic, aggregate consumption growth is constant, and investors have an intertemporal elasticity of substitution equal to one.

We conclude this introduction with two remarks on some broader implications of our model. First, indeterminacy and equilibrium multiplicity are helpful devices to illustrate in the starkest possible way the interplay between the wealth distribution and discount rate changes, which is a feature of heterogeneous-agent models (and are absent in representative agent models). While we use extrinsic shocks to drive fluctuations in asset prices, and illustrate the mechanisms at play, the feedback effects that we identify are present whether the uncertainty in the model is driven by fundamental or extrinsic shocks.

Second, while in this paper we don't consider welfare and policy implications, we would like to remark that the indeterminacy identified in our model is distinct from the type of indeterminacy that routinely arises in macroeconomic models with endogenous interest rate rules. Given the rapidly growing macroeconomics literature on the effects of macroeconomic policy in heterogeneous-agent setups, we believe that the indeterminacies that arise in our model provide a new channel by which stabilization policy could have valuable effects.

1.1 Relation to the literature

The paper relates to various strands of the literature. One strand generates multiple equilibria and indeterminacy, through a variety of mechanisms. Of these, the main ones are a) bubbles (or money) in OLG economies, b) increasing returns to scale and production externalities, and c) portfolio constraints.[2]

[2]The survey Benhabib and Farmer (1999) lists the different mechanisms that lead to indeterminacies.

Our paper is closest to a).[3] As mentioned earlier, we dispense with the requirement that a bubble exist in the economy to obtain indeterminacy, thus sidestepping the empirical challenges associated with the requirements for the existence of a bubble.[4,5] Our mechanism does not involve increasing returns to scale or production externalities. While our model features creative destruction, this creative destruction is useful only in causing redistribution; it does not affect aggregate output or aggregate productivity, which are both exogenous. Specifically, if we linked the different generations altruistically — thus restoring a representative agent economy — the indeterminacy would disappear, unlike the indeterminacies that arise in endogenous growth models. Our mechanism also does not include portfolio constraints for any agent that has already joined the market.

Since the focus of the paper is on reconciling volatile asset prices with non-volatile macroeconomic aggregates (consumption, dividends, etc.), the paper naturally belongs to the literature on macro-asset pricing. The leading examples in this literature are the representative agent frameworks of Campbell and Cochrane (1999), Bansal and Yaron (2004), and Barro (2006), which abstract from redistribution. Just as these models, our paper strives to qualitatively and quantitatively address not just the equity premium puzzle, but an entire host of other asset pricing facts, such as the low and non-volatile interest rate, the predictability patterns of dividend, consumption and excess returns, etc. Given the usage of recursive preferences, we are closer to Bansal and Yaron (2004), but with some important differences: a) in our model aggregate consumption growth is constant, b) we do not require an intertemporal elasticity of substitution above one — our results are derived for an IES equal to one, and would continue to hold for an IES less than one, and c) we do not require separate volatility shocks to obtain countercyclical risk premiums.

Because of our usage of a perpetual youth framework, we relate to papers using overlapping generations to explain asset pricing fluctuations driven by fundamental shocks.[6] We differ from

[3]Tirole (1985), Blanchard and Watson (1982), and Santos and Woodford (1997) are some seminal contributions on bubbles. See Brunnermeier (2008) for a recent survey. Cass and Shell (1983) is the seminal contribution on indeterminacy.

[4]Abel et al. (1989) tests for dynamic inefficiency, which is a condition for the existence of bubbles, and find that the evidence points to a dynamically efficient economy, which precludes the possibility of bubbles. Giglio et al. (2016) examine the long term discounting of real estate in England and Hong Kong and find no evidence of a rational bubble.

[5]We should note here that there are several different variants of what the literature labels as "bubbles." For instance, Harrison and Kreps (1978) and Scheinkman and Xiong (2001) discuss bubbles in dynamic environments with disagreement and short sale constraints. These bubbles reflect resale premiums and are determinate. These bubbles differ from the "rational" bubbles that we discuss later in the paper, in that rational bubbles require that an investor with rational beliefs earn the required rate of return from his or her investment in the bubble.

[6]Indicative examples include Constantinides et al. (2002), Abel (2003), Gomes and Michaelides (2005), Gârleanu and Panageas (2015), Gârleanu and Panageas (2019), Gomez (2017), Maurer (2017), and Schneider (2017). See Panageas (2019) for a survey.

these papers by focusing on the feedback between redistribution and interest rates and showing that it can lead to self-fulfilling expectations. Farmer (2018) considers the asset pricing implications of indeterminacy in a model with money. Başak (2000), Başak et al. (2008), DeMarzo et al. (2008), Gârleanu et al. (2015), Barlevy and Veronesi (2000), Miao and Wang (2018), and Zentefis (2019) are other examples of dynamic asset pricing frameworks that can lead to indeterminacies, albeit for different reasons than the ones we highlight in this paper.

2 Model

2.1 Consumers

Time is continuous. Each agent faces a constant hazard rate of death $\lambda > 0$ throughout her life, so that a fraction λ of the population perishes at each instant. A new cohort of mass λ is born per unit of time, so that the total population remains at $\lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} ds = 1$.

Consumers maximize

$$E_s \int_s^{\infty} e^{-\rho(t-s)} \log\left(c_{t,s}\right) dt, \tag{1}$$

where s is the time of their birth and t is calendar time. The assumption of logarithmic preferences, which implies an inter-temporal elasticity of substitution (IES) equal to one, facilitates the exposition, but is inessential. In Section 5 we consider recursive preferences with IES equal to one, but a different risk aversion. In Appendix C we extend the baseline model to allow for an IES different from one. Agents have no bequest (or gift) motives for simplicity.

2.2 Endowments

The total endowment of the economy is denoted by Y_t and evolves exogenously according to

$$\frac{\dot{Y}_t}{Y_t} \equiv g$$

with $g > 0$. To sharpen our results, we abstract from aggregate uncertainty about the endowment.

All agents have the same preferences, but their initial endowments differ. Specifically, at birth agents can be of two types, "human capitalists" or "entrepreneurs," depending on the nature of their endowment. The time-t fraction of human capitalists in every arriving cohort is $1 - \varepsilon_t$, while

the fraction of entrepreneurs is ε_t. These fractions are not constant, but cohort-dependent and endogenous. Before analyzing the determination of ε_t, we describe the endowments of the two types of agents.

2.3 Human capital

Letting $l_{t,s} \equiv \lambda (1 - \varepsilon_s) e^{-\lambda(t-s)}$ denote the measure of human capitalists that were born at time s and have survived to time t, the per-capita endowment $w_{t,s}$ of a human capitalist born at time $s \leq t$ is given by

$$w_{t,s} \equiv \frac{(1-\alpha) Y_t \left(\delta^l + g\right) e^{-(\delta^l+g)(t-s)}}{l_{t,s}}, \tag{2}$$

where $\alpha \in (0,1)$ and $\delta^l > -g$ are constants. The constant δ^l captures the obsolescence rate of a given cohort's human capital over time.

Aggregating over all cohorts in the economy implies that the aggregate human capital proceeds are a constant fraction $(1-\alpha)$ of output:

$$\int_{-\infty}^{t} w_{t,s} l_{t,s} ds = Y_t (1-\alpha) \left(\delta^l + g\right) \int_{-\infty}^{t} e^{-\left(\delta^l+g\right)(t-s)} ds = (1-\alpha) Y_t. \tag{3}$$

2.4 Risky entrepreneurship

At the time of their birth s, a fraction $\bar{\varepsilon} < 1$ of arriving agents have the ability to become entrepreneurs. These potential entrepreneurs create a firm and introduce it into the stock market at time s. They use the proceeds from the sale of this firm to finance their life-time consumption. For the purposes of this paper, a firm is just a dividend stream, akin to a "Lucas tree."

We index potential entrepreneurs by $i \in [0, \bar{\varepsilon}]$. They are faced with two choices at birth. The "safe" choice is to create a new company that produces dividends

$$D_{t,s}^{(i)} = \psi \alpha Y_t e^{-\int_s^t (\delta_u^d + g) du}, \tag{4}$$

for times $t \geq s$, where $\psi > 0$ and $\delta_t^d > 0$ reflects an obsolescence process that we specify shortly. The alternative, "risky," choice is to introduce a company that is successful with probability $\pi \in (0,1)$.

Specifically, it produces dividends

$$D_{t,s}^{(i)} = \begin{cases} \xi^i \alpha Y_t e^{-\int_s^t (\delta_u^d + g) du} & \text{with probability } \pi \\ 0 & \text{with probability } 1 - \pi \end{cases} \tag{5}$$

at all times $t \geq s$. Here, ξ^i is entrepreneur specific and known to the entrepreneur before she makes her choice. In case of success, the value ξ^i is common knowledge in the economy. Without loss of generality, we assume that $\xi : [0, \bar{\varepsilon}] \to R^+$ is a decreasing function, i.e., entrepreneurs with a low index i can create a more profitable firm than the ones with a high index i.

If the firm ends up being worthless, the entrepreneur becomes a human capitalist for the remainder of her life, so as to finance a positive consumption stream. One possible interpretation is that the "human capitalists" are scientists who choose to either work for the existing firms, or create their own companies. In the latter case, in the event of failure they can rejoin the workforce.

The choice between the safe and the risky option happens once, at birth, and the uncertainty associated with the risky choice is resolved immediately and publicly before the firm's introduction to the market. Since agents are members of a continuum, they take all prices as given when making the choice between the safe and the risky option.

2.5 Aggregate dividends and displacement

Throughout we let $D_{t,s}$ denote the total time-t dividends accruing to firms born at time s. We now derive an expression for the total dividends accruing to newly born firms ($D_{t,t}$). If a measure $\zeta_t < \bar{\varepsilon}$ of entrepreneurs chooses the risky choice, then aggregating gives

$$D_{t,t} = \int_0^{\bar{\varepsilon}} D_{t,t}^{(i)} di = \alpha Y_t \left(\pi \int_0^{\zeta_t} \xi^i di + (\bar{\varepsilon} - \zeta_t) \psi \right). \tag{6}$$

We make the following implicit assumption on model primitives.

Assumption 1 *It holds that*

$$\delta_t^d + g = \pi \int_0^{\zeta_t} \xi^i di + (\bar{\varepsilon} - \zeta_t) \psi. \tag{7}$$

Assumption 1 states that the proportion of aggregate dividends accounted for by new firms equals the depreciation rate of the dividends of existing firms plus the aggregate growth rate. This

assumption captures — in a stylized way — the popular idea that entry of new firms is a source of disruption to existing firms ("creative destruction"). This rivalry between entering and incumbent firms is a key feature of several modeling frameworks. Indicatively, we refer to the seminal papers Romer (1990) and Aghion and Howitt (1992).[7] Here we go farther and make the starker assumption that entry does not affect aggregate growth, in order to isolate the redistribution mechanism, on which the paper focuses. In Appendix D.1 we show that modifying the model to allow entry that promotes aggregate growth does not affect the results of the paper.

One way to highlight the redistributive aspect embedded in Assumption 1 is to combine equation (7) with equations (4) and (5) to infer that the total time-t dividends produced by firms born at time s are given by

$$D_{t,s} = \alpha \left(\delta_s^d + g\right) Y_t e^{-\int_s^t (\delta_u^d + g) du}. \tag{8}$$

Consequently, defining aggregate dividends as $D_t^A \equiv \int_{-\infty}^t D_{t,s} ds$, differentiating D_t^A with respect to time t yields

$$\frac{\dot{D}_t^A}{D_t^A} = \underbrace{\frac{\int_{-\infty}^t \dot{D}_{t,s} ds}{D_t^A}}_{=-\delta_t^d} + \underbrace{\frac{D_{t,t}}{D_t^A}}_{=\delta_t^d + g}. \tag{9}$$

Equation (9) decomposes the change in aggregate dividends into two components. The first component is the proportional decline in the dividends of existing firms $\left(-\delta_t^d\right)$, while the second component captures the proportion of aggregate dividend growth due to the newly arriving firms $\left(\delta_t^d + g\right)$. Hence, δ_t^d determines the redistribution of dividend income between existing and arriving firms.

A further implication of Assumption 1 is that aggregate dividends constitute a fraction α of output ($D_t^A = \alpha Y_t$) for any path of δ_t^d,[8] as is required to account for the entire output Y, which

[7] Indeed, a mathematically equivalent formulation of our model would be to assume that the dividends of individual firms don't depreciate with certainty, but instead that every existing firm faces an instantaneous hazard rate δ_t^d of being displaced by an entrant.

[8] Aggregating equation (8) across all firms gives

$$\int_{-\infty}^t D_{t,s} ds = \alpha Y_t \int_{-\infty}^t \left(\delta_s^d + g\right) e^{-\int_s^t (\delta_n^d + g) dn} ds = \alpha Y_t. \tag{10}$$

A technical requirement is that $\int_{-\infty}^t (\delta_s^d + g) ds = \infty$ a.s. Our proposed δ_s, a "regular" diffusion on $(-g, \bar{\delta})$ for some $\bar{\delta} > -g$, satisfies this condition.

accrues as either wages or dividends.

2.6 Markets

Markets are dynamically complete. Investors can trade in instantaneously maturing riskless bonds in zero net supply, which pay an interest rate r_t. Consumers can also trade claims on all existing firms (normalized to unit supply). Finally, investors can access a market for annuities through competitive insurance companies as in Blanchard (1985). We refer to Blanchard (1985) for details of this annuity market. Briefly, the presence of annuities allows agents to receive an income stream of $\lambda W_{t,s}$ per unit of time, where $W_{t,s}$ is their financial wealth. In exchange, the insurance company collects the agent's financial wealth when she dies. Entering such a contract is optimal for all agents, given the absence of bequest motives. The budget constraint of a human capitalist is

$$dW_{t,s} = (r_t + \lambda) W_{t,s} dt + w_{t,s} dt + \int_{-\infty}^{t} \theta_{t,s} (dP_{t,s} + D_{t,s} dt - r_t dt) ds, \tag{11}$$

where $W_{t,s}$ is a consumer's wealth, $P_{t,s}$ is the value of the representative firm of vintage s, and $\theta_{t,s}$ is the number of shares of each company. For a worker, $W_{t,t} = 0$. An entrepreneur's dynamic budget constraint is identical, except that the term $w_{t,s}$ is replaced by zero and the initial wealth $W_{t,t}$ is given by the value of the firm that the entrepreneur creates.

2.7 Equilibrium

The equilibrium definition is standard. We look for consumption processes $c_{t,s}$, asset allocations $\theta_{t,s}$, asset prices $P_{t,s}$, a process of entrepreneurial risk choice $\zeta_t \in [0, \bar{\varepsilon}]$, and an interest rate r_t such that a) consumers maximize (1) subject to (11); b) newly-born entrepreneurs optimally choose whether to take the risky or the riskless choice; c) the goods market clears, i.e., $\lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} c_{t,s} ds = Y_t$; and d) assets markets clear, i.e., $\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} \theta_{t,s} ds = 1$ and $\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} (W_{t,s} - \theta_{t,s} P_{t,s}) ds = 0$.

3 Baseline Model: Solution and Analysis

3.1 Deterministic equilibria

Our model features no fundamental shocks. In this section we derive a deterministic equilibrium, which helps us highlight the presence of multiple equilibria. We also use this simple model to

introduce stochastic equilibria capturing shifts in the manner in which investors coordinate expectations.

3.1.1. Definitions and ancillary results

We start by introducing some notation. First, the effective discount rate:

$$\beta \equiv \lambda + \rho. \tag{12}$$

Second, we use $q^d_{t,s}$ to denote the "price-dividend ratio," that is, the ratio of the present value of the dividend stream $D_{u,s}$ to the current level $D_{t,s}$:

$$q^d_{t,s} \equiv \frac{E_t \int_t^\infty e^{-\int_t^u r_v dv} D_{u,s} du}{D_{t,s}}. \tag{13}$$

We note that $q^d_{t,s}$ is independent of s, since $\frac{D_{u,s}}{D_{t,s}}$ is not a function of s. Accordingly, we write q^d_t rather than $q^d_{t,s}$. In a similar fashion we define the ratio of the present value of human capital to current wages as

$$q^l_{t,s} \equiv \frac{E_t \int_t^\infty e^{-\int_t^u (r_v+\lambda) dv} w_{u,s} du}{w_{t,s}}. \tag{14}$$

Similar to q^d_t, $q^l_{t,s}$ does not depend on s, and we will write q^l_t.

At this stage, the expectation operator in equations (13) and (14) appears superfluous (since we are constructing a deterministic equilibrium), but it will become useful later.

To prepare for the analysis of the model, and economize on notation, we note the following relation between q^d_t and q^l_t.

Lemma 1 *In any bubble-free equilibrium*

$$(1-\alpha)\, q^l_t = \frac{1}{\beta} - \alpha q^d_t. \tag{15}$$

Equation (15) is intuitive. To derive it, observe that the sum of present value of all dividend income accruing to existing firms $\left(q^d_t \alpha Y_t\right)$ plus the present value of all earnings accruing to existing agents $\left(q^l_t (1-\alpha) Y_t\right)$ multiplied by the consumption-to-wealth ratio β should equal aggregate

consumption C_t:

$$\beta \left(q_t^d \alpha Y_t + q_t^l (1-\alpha) Y_t \right) = C_t. \tag{16}$$

Recognizing that in equilibrium $C_t = Y_t$ leads to (15).

The linear relation between q_t^d and q_t^l captured by (15) implies that we need only characterize the equilibrium behavior of q_t^d; the behavior of q_t^l is determined by that of q_t^d. In light of this observation, from now on we will use the simpler notation q_t interchangeably with q_t^d to refer to the price-dividend ratio.

3.1.2. Solution

In this section we use the Euler equation along with goods market clearing to derive an expression for the equilibrium interest rate and a differential equation for the equilibrium price-dividend ratio q_t. The main goal of the section is to show that the differential equation characterizing the dynamics of q_t has a stable steady state and multiple transition paths that lead to this steady state.

The first step of the analysis is to note that time differentiation of (13) implies that q_t satisfies the familiar asset pricing equation

$$\frac{\dot{q}_t}{q_t} - \delta_t^d + \frac{1}{q_t} = r_t, \tag{17}$$

subject to a standard transversality condition. Equation (17) is an indifference relation between stocks and bonds that needs to hold when risk premiums are absent. It states that the total return on a stock, comprised of the expected capital gain $\frac{\dot{q}_t}{q_t} - \delta_t^d$ plus the dividend yield $\frac{1}{q_t}$, should equal the interest rate r_t.

The second step towards characterizing the dynamics of q_t is to determine the equilibrium interest rate r_t by using the Euler equation and goods market clearing. Letting $c_{t,s}$ denote the time-t consumption of a consumer born at time s, the Euler equation (in the presence of annuities) implies

$$\frac{\dot{c}_{t,s}}{c_{t,s}} = -\left(\rho - r_t\right). \tag{18}$$

Using the definition of aggregate consumption $C_t = \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} c_{t,s} ds$ together with (18) implies

$$\begin{aligned}\dot{C}_t &= -\lambda C_t + \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} \dot{c}_{t,s} ds + \lambda c_{t,t} \\ &= -\left(\lambda + \rho - r_t\right) C_t + \lambda c_{t,t}. \end{aligned} \tag{19}$$

Market clearing implies $C_t = Y_t$ and accordingly $\dot{C}_t = g C_t$. Therefore (19) leads to

$$r_t = \rho + g + \lambda - \lambda \frac{c_{t,t}}{C_t}. \tag{20}$$

Equation (20) captures the main departure of our model from a typical representative-agent model. In a representative-agent model with logarithmic preferences, the Euler equation implies that the interest rate is given by $\rho + g$, the sum of the discount rate ρ and the consumption growth g of the "representative agent," whose consumption coincides with aggregate consumption. In our model, the Euler equation (18) applies at the level of an individual, but her consumption growth differs from aggregate consumption growth. Specifically, the consumption growth of a fixed cohort member is given by the aggregate consumption growth rate (g) plus the consumption share no longer consumed by perishing agents (λ) minus the consumption share accruing to the incoming cohort ($\lambda \frac{c_{t,t}}{C_t}$). This explains the presence of the term $\lambda - \lambda \frac{c_{t,t}}{C_t}$ on the right hand side of equation (20).

To solve for the interest rate, we need an expression for $\lambda \frac{c_{t,t}}{C_t}$. Imposing the intertemporal budget constraint at the time of a consumer's birth, we have the following result.

Lemma 2 *The newly-born agents' consumption is given by*

$$\frac{c_{t,t}}{C_t} = \frac{\beta}{\lambda} \left((1-\alpha)\left(\delta^l + g\right) q_t^l + \alpha(\delta_t^d + g) q_t \right). \tag{21}$$

Equation (21) is intuitive. It states that the per-capita consumption of newly-born agents equals the consumption-to-wealth ratio for an investor with unit elasticity of substitution (β) multiplied by the per-capita value of total (thus, non-traded and traded) wealth, which is given by the expression inside the outer parentheses.

Combining (20) with (21) and using (15) gives an expression for the equilibrium interest rate:

$$r_t = \beta - \delta^l - \beta\alpha(\delta_t^d - \delta^l)q_t. \tag{22}$$

Letting

$$\eta_t \equiv \delta_t^d - \delta^l, \tag{23}$$

combining (17) with (22), and re-arranging leads to the differential equation

$$\dot{q}_t = (\beta + \eta_t)\, q_t - \beta\alpha\eta_t q_t^2 - 1. \tag{24}$$

Equation (24) describes the dynamics of q_t up to the determination of η_t. The value of η_t is endogenous and, by Assumption 1, depends on the fraction ζ_t of newly arriving entrepreneurs who choose the risky over the riskless option. The next lemma provides a relation between η_t and q_t.

Lemma 3 *The measure ζ_t of agents choosing the risky options is a (weakly) decreasing function of $\frac{q_t}{q_t^l}$. Therefore, $\eta_t = \eta(q_t)$ with $\eta'(q_t) \leq 0$.*

To understand Lemma 3, note that an entrepreneur is risk averse and is concerned more with the possibility that the risky project may fail (the downside) than with the prospect of its success (the upside). In the case of failure, the entrepreneur still collects the proceeds of her human capital. Therefore, a relatively high value of human capital q_t^l (which is associated with a low value of q_t by Lemma 1) makes the risky choice comparatively more attractive.

If we think of our "human capitalists" as scientists, then Lemma 3 implies that a higher value of the present value of wages for scientists leads to more innovation and hence displacement of older firms because of the increased attractiveness of the fall back option to become a scientist.

Remark 1 *In an extension to the model that we present in the appendix (Section D.2), we argue that it is inessential for the model that all workers be scientists who stand to benefit from creative destruction. Indeed, all our conclusions go through if the workers are heterogeneous with some "low-skilled" workers suffering from displacement just like existing firms. We also show that it is inessential for all firms to suffer from displacement.*

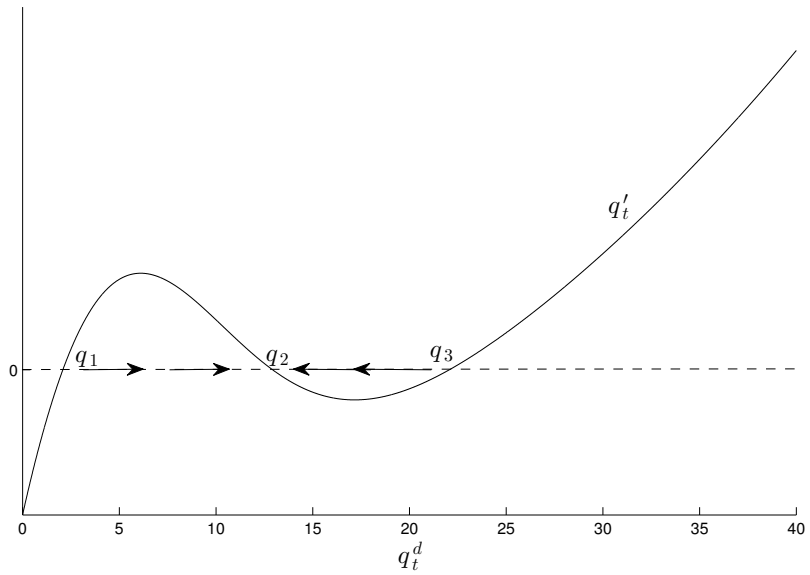

Figure 1: An illustration of Proposition 1.

3.1.3. Stable steady state and multiplicity of transition paths

Lemma 3 shows the existence of a decreasing function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta_t = \eta(q_t)$. We proceed to write equation (24) compactly as

$$\begin{aligned} \dot{q}_t &= A(q_t) \\ &\equiv (\beta + \eta(q_t))\, q_t - \beta\alpha\eta(q_t)\, q_t^2 - 1. \end{aligned} \tag{25}$$

Equation (25) is the key equation of the paper, since it characterizes the dynamic behavior of the price-to-dividend ratio. Inspection of (25) shows that the dynamics of q_t depend crucially on the shape of the function η. The assumptions on the distribution of initial productivity ξ^i determine how fast η declines. (The proof to Proposition 1 in the appendix provides a mapping between any given downward sloping η and a distribution of ξ^i and associated parameter ψ.) The following proposition shows that η can be chosen so that the differential equation $\dot{q}_t = A(q_t)$ has a stable steady state.

Proposition 1 *For any three real numbers $0 < q_1 < q_2 < q_3 < \frac{1}{\alpha\beta}$, there exist parameters under which q_i, $i \in \{1, 2, 3\}$, are roots of $A(q)$ with $A'(q_1) > 0$, $A'(q_2) < 0$, and $A'(q_3) > 0$.*

Figure 1 illustrates Proposition 1. The figure shows that that $\dot{q}_t$ is positive between q_1 and

q_2 and negative between q_2 and q_3. An immediate implication is that the dynamical system (25) has a stable steady state. Any initial value $q_{t_0} \in (q_1, q_3)$ is associated with a different equilibrium transition path to the steady state q_2. Interestingly, all of these paths constitute different, perfect-foresight equilibria and the economic structure cannot rule out any of them.

The presence of multiple equilibrium paths ("indeterminacy") is an uncommon property for a neoclassical model, especially one that features neither bubbles, nor increasing returns to scale in production. We next explain the economic intuition behind this indeterminacy. In the next section we discuss the relation between our result and the indeterminacy obtained in similar OLG models featuring bubbles.

The source of indeterminacy is investors' self-fulfilling expectations about future discount rates and the effects that these expectations have on redistribution between existing and arriving cohorts. To give an example, suppose that at time t investors expect to approach the steady state q_2 through an increasing path of discount rates, i.e., that discount rates on the transition path will be lower than at the steady state. This anticipation of low interest rates on the transition path raises the current value of human capital and stimulates the creation of new firms, which displace the dividends of old companies. The cohort of agents born at time t unambiguously benefits, since the value of their human capital increases and their cohort benefits from increased firm creation. Since aggregate consumption is given at time t and the consumption-to-wealth ratio is fixed, the increased wealth of the younger cohorts implies that the arriving cohort appropriates a larger share of aggregate consumption ($\frac{c_{t,t}}{C_t}$), resulting in a lower consumption growth rate for older cohorts. Finally, only older cohorts are marginal in asset markets, and therefore their lower consumption growth rate is reflected in lower interest rates (equation (20)), confirming the expectations of low interest rates.

One ingredient of the above argument is that the entry rate of new firms must respond sufficiently to variation in the valuation ratio — more precisely, the function η needs to be decreasing sufficiently strongly with q. The proof of the proposition in the appendix provides the exact condition.

Remark 2 *The requirement that η_t be negatively related to q_t is special to the simplifying assumption that the IES is equal to one. As we discuss in Appendix C, the relation between η_t and q_t can be positive if the IES is below one, while the equilibrium remains indeterminate. When the IES is below one, the wealth-to-consumption ratio is itself time-varying, and hence the value of human*

and firm capital don't have to offset each other, as they do in the case of unitary IES (Lemma 1). In particular, depending on parameters, the ratio of firm-to-human capital (q_t^d/q_t^l) *may decline — which, by Lemma 3, implies that* η_t *increases — while both* q_t^d *and* q_t^l *increase. Proposition 6 in Appendix C provides a formal result.*

3.2 Stochastic "sunspot" equilibria

An immediate implication of the multiplicity identified in the baseline model is the potential for so-called "sunspot" equilibria, i.e., stochastic equilibria where the source of uncertainty is not about fundamentals (preferences, endowments, etc.), but rather reflects random fluctuations in agents' perceptions about the equilibrium path that the economy will follow.

To construct such equilibria, we introduce a standard brownian motion B_t. This Brownian motion reflects random "noise" that is extrinsic to the economy; however, everyone understands (and knows that everyone else also understands) that this noise acts as a coordination device for investor expectations (e.g., speeches, articles, a perception of market "sentiment,"etc.).

The next proposition states the existence of equilibria whereby an understanding among investors that the noise B_t is useful in coordinating expectations ends up becoming self-fulfilling, in the sense that it affects both asset-price dynamics and equilibrium consumption allocations.

Proposition 2 *For* q_1 *and* q_3 *as in Proposition 1, take an interval* $[q^{min}, q^{max}] \subset (q_1, q_3)$ *with* $A\left(q^{min}\right) > 0$ *and* $A\left(q^{max}\right) < 0$*. Further, choose a bounded function* $\sigma : (q^{min}, q^{max}) \to R^+$ *with the properties* $\sigma(x) > 0\ \forall x$ *and*

$$\lim_{q \to q^{max}} \left(\frac{\sigma^2(q)}{q^{max} - q} \right) < 2\left|A\left(q^{max}\right)\right|, \quad \lim_{q \to q^{min}} \left(\frac{\sigma^2(q)}{q - q^{min}} \right) < 2A\left(q^{min}\right). \tag{26}$$

Then there exists an equilibrium whereby the equilibrium stochastic process for q_t *is given by the diffusion*

$$dq_t = A(q_t)\, dt + \sigma(q_t)\, dB_t. \tag{27}$$

In such an equilibrium q_t *possesses a stationary distribution, equation (15) continues to hold, and* r_t *continues to satisfy (22).*

Proposition 2 ensures the existence of sunspot equilibria, i.e., equilibria where the process q_t

is stochastic and driven by the noise process B_t. Inspection of equation (27) shows that these processes have the same drift $A(q_t)$ as in the deterministic case, and in addition feature a volatility $\sigma(q_t)$, which is essentially arbitrary (up to satisfying the technical condition (26), which ensures stationarity).

To see intuitively why such equilibria exist, we start by noting that, even if an equilibrium is stochastic (in the sense that q_t is stochastic), the consumption of existing investors — who are marginal for pricing assets — must be locally deterministic.[9] Accordingly, equations (18) and (20) continue to hold, as do Lemmas 1, 2, and 3. The main equation that needs to be modified in the presence of noise is the asset-pricing equation (17), which now includes a diffusion term. As the consumption of existing agents has no quadratic variation, there is no risk premium in this economy. Consequently, equation (17) continues to describe the drift of dq_t. (As we explain later, this implication no longer holds when investors have recursive preferences.)

4 Relation to Rational Bubbles

A novel aspect of the indeterminacy in the present model is that it does not require a "bubble" in the economy. Indeed, the steady state that we consider can feature an interest rate that is higher than the growth rate of output, a situation in which no rational bubble can exist.

In this section, we discuss the relation between our mechanism for indeterminacy and the one based on bubbles. In the context of our model we identified three elements that are responsible for indeterminacy: a) the existence of multiple assets (firm value and human capital), which react differently to changes in discount rates; b) the fact that existing and arriving cohorts of agents are differentially endowed with these assets; and c) the fact that only existing agents play a role in price formation (arriving agents can only participate in markets after their birth). In this section, we argue that these same features are responsible for indeterminacy in models with bubbles, except that the role played by the multiple assets in our model is played by the presence of "fundamental values" and "bubbles" as two distinct stores of value in bubble-based models.

We proceed now to explain these statements in more detail. To that purpose, we modify our baseline model by (i) eliminating features that are necessary only to obtain indeterminacy in the absence of bubbles and (ii) dropping the no-bubble (transversality) assumption, so that

[9] This property follows from the fact that aggregate consumption is deterministic, while newly born agents arrive at a finite rate.

equation (15) need no longer hold.

Specifically, entrepreneurs' entry rate is exogenous, and further we assume in this section that $\delta^d = \delta^l =: \delta$. It follows, using definitions (13) and (14), that $q^d_{t,s} = q^l_{t,s} =: q_t$. As in Section 3, the Euler equation (18) and the goods-market clearing equation (19) imply that the interest rate continues to be given by

$$r_t = \rho + g + \lambda - \lambda \frac{c_{t,t}}{C_t}. \tag{28}$$

Assuming, further, that the value of a newly arriving firm equals the present value of its dividends, the assumption $\delta^l = \delta^d$ (and accordingly $q^d_t = q^l_t =: q_t$) together with Lemma 2 leads to

$$\frac{c_{t,t}}{C_t} = \frac{\beta}{\lambda}(g + \delta) q_t. \tag{29}$$

Equation (17) remains unchanged, and substituting (29) into (28) and then into (17) gives the Riccati equation

$$\dot{q}_t = (\beta + g + \delta)\, q_t - \beta(g + \delta) q_t^2 - 1. \tag{30}$$

It is noteworthy that equation (30) is not just a special case of equation (25) with $\delta^l = \delta^d$. The difference is that, in deriving equation (25), we imposed Lemma 1, which assumes the absence of bubbles, whereas in deriving equation (30) we did not. (If we were to exclude equilibria with rational bubbles on any asset, then Lemma 1 would hold, and since $\delta^l = \delta^d$, the only solution for the price-dividend ratio would be $q_t = q^l_t = q^d_t = \frac{1}{\beta}$ for all t.)

Since the right-hand side of (30) is quadratic in q_t, there are two values of q_t such that $\dot{q}_t = 0$ (candidate steady states). They are given by $q^* = \frac{1}{g+\delta}$ and $q^{**} = \frac{1}{\beta}$. In these steady states, equations (28) and (29) imply that the interest rate would be $r^* = g$ and $r^{**} = \beta - \delta$, respectively.

The nature of equilibria in this economy rests on whether β is larger than $g + \delta$ or not. As Proposition 3 below shows, if $\beta > g + \delta$ then the economy cannot feature bubbles, and the equilibrium value of q_t is unique. By contrast, if $\beta < g + \delta$, then there are two steady states, one of which is stable. To facilitate the statement of the proposition, we let P_t denote the aggregate value

of financial wealth and b_t the value of the bubble:

$$P_t \equiv \left(\frac{1}{\beta} - (1-\alpha)\, q_t\right) Y_t \tag{31}$$

$$b_t \equiv P_t - \alpha q_t Y_t = \left(\frac{1}{\beta} - q_t\right) Y_t, \tag{32}$$

where we used the fact that, due to the logarithmic preferences, the aggregate wealth in the economy equals $\frac{C_t}{\beta} = \frac{Y_t}{\beta}$.

Proposition 3 *(i) If $\beta > g + \delta$, then the unique equilibrium features $q_t = q^{**} = \frac{1}{\beta}$ and $b_t = 0$. (ii) If $\beta < g + \delta$, then both q^* and q^{**} correspond to steady-state equilibria. Furthermore, to any initial value $q_0 \in [q^*, q^{**}]$ there corresponds an equilibrium, along whose path it holds that $\dot{b}_t = r_t b_t$ for all $t > 0$. If $q_0 > q^*$, then $\lim_{t\to\infty} q_t = q^{**}$ and $\lim_{t\to\infty} \frac{b_t}{Y_t} = 0$.*

In case (i) the equilibrium features the property $r > g$ in steady state.[10] By a well known argument (see, for instance, Tirole (1985)) there cannot be any bubbles in this case, since there would be a contradiction: Any bubble has to grow at the rate r_t and would eventually become larger than aggregate financial wealth P_t (which grows at the same rate as output, namely g, by equation (31)). Equation (32) would then imply a negative q_t, contradicting the fact that $q_t = \frac{1}{\beta} > 0$.

In case (ii), $r \leq g$ in either steady state, which allows for indeterminate equilibria. The different equilibrium paths associated with different values q_0 feature different initial values of the bubble, b_0. The bubble grows at the interest rate r_t, and hence investors earn the required rate of return. In all equilibria that emanate from $q_0 \in (q^*, q^{**}]$ the bubble grows more slowly than aggregate consumption C_t (and aggregate market capitalization P_t), since $r_t < g$. An exception is the equilibrium that starts with $q_0 = q^*$, in which $r_t = g$ and the bubble remains a constant fraction of the aggregate economy.

The presence of bubbles introduces multiple equilibria in a way that is similar to the mechanism of the baseline, bubble-less model. There, the total wealth in the economy — the sum of financial wealth and the present value of all wages accruing to existing agents ("human capital") — is determinate and equal to $\frac{C_t}{\beta}$. The indeterminacy arises because the fraction due to each of the two components of wealth is indeterminate. In the presence of bubbles, the total wealth (the sum of

[10] To confirm this use $q_t = \beta^{-1}$ inside (29) and then use the resulting expression for $\frac{c_{t,t}}{C_t}$ inside (28) to conclude that $r_t = \rho + g + \lambda - (g + \delta) = \beta - \delta > g$.

the present value of dividends, human capital, and bubble) is determinate (and equal to $\frac{C_t}{\beta}$), while the composition is indeterminate — because the bubble is.

An intuition analogous to the one that we presented in Section 3 helps explain the multiplicity of equilibrium paths. Take, for instance, two initial values of q_0, say q_0^A and q_0^B with $q^* < q_0^A < q_0^B < q^{**}$. Both initial values are consistent with a specific path of rational (self-fulfilling) expectations of interest rates. In case A, as compared to case B, investors expect a longer transition path with higher initial interest rates and a higher value of the initial bubble. This expectation becomes self-fulfilling because the higher initial interest rates and the higher initial value of the bubble shift wealth towards the existing investors, thus lowering the fraction of consumption accruing to incoming cohorts, and increasing the consumption growth rate of existing cohorts. Because the Euler equation of existing agents needs to hold, this increase in their consumption growth rate does indeed lead to a higher interest rate in equilibrium, confirming expectations.

The main difference between our baseline model and the one in this section is that the former does not rely on bubbles, which may in fact not even be possible. As a consequence, the parameter restriction $\beta > g + \delta$ and the associated implication that $r < g$ are not necessary for indeterminacy.

We conclude with a remark. In this section we focused on equilibria in which arriving assets are priced at their fundamental value. In Appendix B we extend the model to allow arriving agents to be endowed with new bubbles. An interesting aspect of this extension is that bubbles never perish, but exist in steady state.

5 Recursive Preferences and Risk Premiums

The sunspot equilibria of Proposition 2 imply that in this model one has substantial freedom to specify the volatility process for asset prices so as to match the high empirical volatility of asset returns. An unattractive feature of such equilibria, however, is that there can be no risk premiums if investors have expected utility preferences. The consumption processes of all agents are locally deterministic processes (i.e., have no diffusion component); therefore, despite volatile asset prices, the risk premium remains zero.

In this section we show that risk premiums are non-zero when investors have recursive Epstein-Zin-Weil (EZW) utilities. Intuitively, with EZW utilities risk premiums are affected not only by the covariance between asset returns and instantaneous consumption growth, but also by the covari-

ance between asset returns and long-run consumption growth. In this model, while instantaneous consumption growth is locally deterministic, the consumption growth of a given cohort of investors integrated over a period of time is stochastic. This stochastic nature of long run consumption growth gives rise to positive risk premiums.

We keep the presentation in this section intentionally concise, because the introduction of recursive preferences does not affect any of the key insights of the model beyond generating a risk premium. Specifically, the model is the same as in Section 2, with one modification: while agents' intertemporal elasticity of substitution continues to equal one, their risk aversion may take any value. Specifically, we assume the investors' instantaneous utility flow is no longer given by $\log(c_{t,s})$ but rather by the aggregator

$$f(c_{t,s}, V_{t,s}) = (1+\gamma V_{t,s})\left(\log c_{t,s} - \frac{\beta}{\gamma}\log(1+\gamma V_{t,s})\right), \tag{33}$$

where $V_{t,s}$ is a consumer's value function and $\gamma < \beta$ controls her risk aversion. Utilities of this form are discussed extensively in Duffie and Epstein (1992) and Schroder and Skiadas (1999). They correspond to the continuous-time limit of Epstein-Zin-Weil utilities with unit elasticity of substitution.

As is highlighted in Schroeder and Skiadas (1999), a convenient transformation is given by $U_{t,s} \equiv \frac{1}{\gamma}\log(1+\gamma V_{t,s})$, resulting in

$$U_{t,s} = E_t \int_t^\infty e^{-\beta(u-t)}\left(\log(c_{u,s})\,du + \frac{\gamma}{2} d[U]_{u,s}\right), \tag{34}$$

where $d[U]_{t,s}$ is the time-t quadratic variation of $U_{t,s}$. Following a common convention in the literature, we will refer to $U_{t,s}$ as anticipated "long run consumption growth."

With these preferences, the log-stochastic discount factor (SDF) $\log(m_t)$ follows the dynamics

$$d\log(m_t) = \gamma(\log c_{t,s} - \beta U_{t,s})\,dt - \beta dt + \gamma dU_{t,s} - d\log c_{t,s}. \tag{35}$$

The key feature of (35) is that risk premiums are associated not only with variations in the current consumption flow; variations in anticipated long run consumption growth $U_{t,s}$ enter the stochastic discount factor as well.

We note that, given the unitary intertemporal substitution, the relation between $q_t^d = q_t$ and q_t^l

is still given by Lemma 1, while the cutoff ζ_t, and consequently η_t, continue to be functions of q_t only.

The following proposition contains the main result of this section.

Proposition 4 *Suppose that investors have preferences of the form (33). Take an interval* $[q^{min}, q^{max}] \subset [q_1, q_3]$ *and a volatility process* $\sigma(q_t)$ *with the same properties as in Proposition 2. Then there exists an equilibrium in which the stochastic process for* q_t *obeys the diffusion*

$$dq_t = \left(A\left(q_t\right) + \kappa\left(q_t\right)\sigma\left(q_t\right)\right)dt + \sigma\left(q_t\right)dB_t, \tag{36}$$

where $\kappa\left(q_t\right)$*, the market price of risk (Sharpe ratio) in this economy, is given by*

$$\kappa\left(q_t\right) = -\gamma Z'\left(q_t\right)\sigma\left(q_t\right) \tag{37}$$

with $Z(q)$ *solving the second-order differential equation*

$$\frac{Z''}{2}\sigma(q)^2 + \left(A(q) - \gamma Z'\sigma(q)^2\right)Z' - \beta Z - \alpha\eta q + \frac{\gamma}{2}(Z')^2\sigma(q)^2 = 0. \tag{38}$$

In equilibrium, q_t^l *continues to obey the relation (15) and* r_t *continues to satisfy (22).*

The equilibria associated with Proposition 4 are sunspot equilibria, similar to the ones of Proposition 2. The only material change is that the asset-price dynamics are now given by (36), with the additional term $\kappa\left(q_t\right)\sigma(q_t) = -\gamma Z'(q_t)\sigma(q_t)^2$ reflecting an equity premium. As we show in the appendix, the term $Z'(q_t)\sigma(q_t)$ can be interpreted as the volatility of an agent's long-run consumption growth, with $Z(q_t)$ determined by the differential equation (38). By (37), the Sharpe ratio κ_t is proportional to the volatility of long run consumption growth.

Intuitively, since the drift of the consumption growth of a marginal agent depends on q_t, and q_t is persistent, a sunspot shock to q_t impacts not just the current consumption drift of the marginal agent, but also future consumption drifts. Thus, the stochastic sunspot shocks impact the long-run consumption growth rate of an agent, and — with recursive preferences — introduce a risk premium, which modifies the required rate of return and hence the drift of q_t in equation (36).

In the next section we illustrate Proposition 4 numerically, by solving the differential equation (38) and evaluating the model's implications for asset price dynamics.

6 Quantitative Evaluation

Tests of macro asset-pricing models exploit the links between observed asset prices and real quantities (consumption, dividends, etc.). These same relations remain testable whether the origin of the fluctuations is fundamental (such as an exogenous productivity shock), or extrinsic, in which case asset prices and real quantities are jointly determined in response to expectational shocks. Therefore, our model is testable like any conventional asset pricing model: Conditional on a choice of preference parameters, and a specification of the stochastic process for the volatility of q_t, which can be disciplined by data, all the remaining quantities and prices are uniquely determined (real interest rate, Sharpe ratio, dividend growth, consumption allocations across cohorts, etc.), and therefore provide a basis for comparison to the data.

In order to carry out this comparison, we fix a set of parameters that have direct data counterparts and choose preference parameters in line with the literature. We then choose the functions $\eta(q_t)$ and $\sigma(q_t)$ to reproduce a realistic range, autocorrelation, and volatility of the log-price dividend ratio.

In terms of parameters that have direct empirical counterparts, we choose $\alpha = 0.33$ for the fraction of output accruing as profits and $g = 0.023$ for aggregate growth. We assume a subjective discount rate of $\beta = 0.027$, equal to the death rate plus one percent to reflect time discounting.[11]

For the human capital profile we set $\delta^l + g = 0.05$. This choice is motivated by simplicity, since it implies that a human capitalists' income is constant over her life.[12]

We specify the functions $\eta(q)$ and $\sigma(q)$ so that $\log(q_t)$ has an autocorrelation and volatility similar to the data. Specifically, we choose

$$\eta(q) = \max\left(\frac{q^{-1} - \beta + 0.082 - 0.03\log(q)}{1 - \beta\alpha q}, -(\delta^l + g)\right). \tag{39}$$

The truncation at $-(\delta^l + g)$ is to ensure a non-negative entry rate of firms. The specification (39) implies that the drift of $d\log(q_t)$ is approximately linear in $\log(q_t)$ and has a realistic mean-reversion.[13] We opt for a variance specification that ensures that q_t has a stationary distribution

[11] In the model death and birth rates are equal for simplicity. In the data the birth rate (plus immigration rate) is 2.7% and the death rate is approximately one percent lower than that. For the determination of β the death rate is the relevant quantity.

[12] By equation (2), if $\delta^l = \lambda$, then $w_{t,s}$ is independent of t. Accordingly, if $g = 0.023$, the choice $\delta^l + g = 0.05$. implies that $\delta_l = 0.027$, which is approximately equal to the birth rate (plus immigration rate) in the data.

[13] The specification (39) inside (24) implies that $\frac{\dot{q}_t}{q_t}$ is linear in $\log(q_t)$. Note that the actual drift of $\log(q_t)$ in the model will not be exactly linear for a multitude of reasons. Equation (24) does not account for a risk premium, a

	Data	Model
Average aggregate consumption growth rate	2.3%	2.3%
Volatility of aggregate consumption growth rate	3.3%	0%
Individual agents' annual volatility of consumption growth		0.60%
Sharpe ratio	0.29	0.24
Stock market volatility	18.2%	16.44%
Equity premium	5.2%	4.63%
Interest rate	2.8%	0.80%
Standard deviation of the interest rate	0.92%	0.60%
Average (log) price-dividend ratio	3.21	2.91
Standard deviation of the log price-dividend ratio	0.27	0.28
Autocorrelation of the log price-dividend ratio	0.89	0.92
Dividend growth volatility	11.2%	8.45%

Table 1: Unconditional moments for the data and the model (both annualized). The data for the average equity premium, the volatility of returns, and the level of the interest rate are from the long historical sample available from the website of R. Shiller (http://www.econ.yale.edu/?shiller/data/chapt26.xls). The volatility of the "real rate" is inferred from the yields of 5-year constant maturity TIPS. The model-implied excess returns, dividend growth, and log-price dividend ratio are adjusted for leverage.

in the range (2,40). Specifically, we choose the simple, linear function $\sigma^2(q) = 1.1 \times (40.9 - q)$ for values of q above 18.9. For values below 18.9 we choose an increasing function that ensures the technical requirements for stationarity of Proposition 4.[14] With this specification of volatility, the model produces a simple, declining, and approximately linear relationship between the level and the volatility of the log-price dividend ratio for the 5%-95% stationary range of values of the log price-dividend ratio as the bottom right graph of Figure 2 shows. Finally, the choice of γ corresponds to a relative risk aversion of 10.

Table 1 provides a comparison between the model-implied unconditional moments and the respective moments in the data. In reporting the results we follow the approach of Barro (2006) to relate the results of our model (which produces implications for an all-equity financed firm) to

convexity adjustment is missing, and the truncation by $-(\delta^l + g)$ may bind. Nevertheless, in the simulation we find that specification (39) implies a drift of $\log(q_t)$ that is approximately linear in $\log(q_t)$ around the stochastic steady state.

[14] The precise specification of the volatility in this range has no major impact on the quantitative results, and several specifications would satisfy the stationarity requirements. For the calibration we choose

$$\sigma^2(q) = q^2 \times \left(w(q)\, 0.26^2 + (1 - w(q))\, 0.02^2\right) \times 1_{\{q \in (2.19, 18.9)\}} + 1.1 \times (40.9 - q) \times 1_{\{q \geq 18.9\}} + 0.255 \times (q - 2.18) \times 1_{\{q \leq 2.19\}},$$

where $w(q) = \frac{\log(q) - 0.78}{2.94} - 0.78$.

the data (which pertain to leveraged equity). Specifically, we use the Modigliani-Miller formula relating the returns of leveraged equity to those of unleveled equity, along with the historically observed debt-to-equity ratio, to report model-implied leveraged returns. (To do so, we set the ratio of levered to unlevered equity returns to be equal to 1.7, similar to Barro (2006)).[15] With this adjustment, the excess returns of levered equity are 1.7 times the excess returns of model-implied unlevered equity, the price-dividend ratio of levered equity is $\frac{1}{1.7}$ times the price-dividend ratio of unlevered equity, and the (per-share) dividend growth of levered equity is adjusted for leverage according to equation (98) in Appendix E. For comparison purposes, Table 3 in the appendix contains the model-implied moments for unlevered equity.

Inspection of Table 1 shows that the model delivers simulated moments that are of similar magnitude to their empirical counterparts. Since the model was calibrated to approximately reproduce the time series properties of the price-dividend ratio in the data, we would like to focus attention on the moments that were not targets of the calibration, specifically the Sharpe ratio, the interest rate, and the equity premium, along with the volatility of the interest rate and the dividend-growth volatility of the market portfolio. We find these moments telling for a simple reason: While the model was calibrated to reproduce the volatility of the price-dividend ratio, it was not calibrated to reproduce the fact that the volatility of the price-dividend ratio in the data seems to be mostly driven by variation in discount rates (rather than dividend growth) arising predominantly from variations in the equity premium (rather than the interest rate). Table 1 shows that the volatility of both the interest rate and the dividend growth of the market portfolio are close to their empirical counterparts.

Figure 2 presents the equity premium, Sharpe ratio, interest rate, and volatility of returns as functions of the log-price dividend ratio of levered equity. We choose the range of the log price-dividend ratio in these graphs to correspond to the 5–95 percentile range of the stationary distribution of the log price dividend ratio in the model. The Sharpe ratio and the equity premium decline as the log price-dividend ratio increases, which suggests that the log price-dividend ratio should be able to predict returns with a negative sign. (We discuss the results of such predictability regressions shortly). The figure also shows that the range of values of the equity premium is substantially larger than the respective range for the interest rate. The low volatility of the interest

[15] A more aggressive choice is to set that parameter as the ratio of consumption growth volatility to dividend growth volatility, which results in a value around 3. See, for instance Bansal and Yaron (2004).

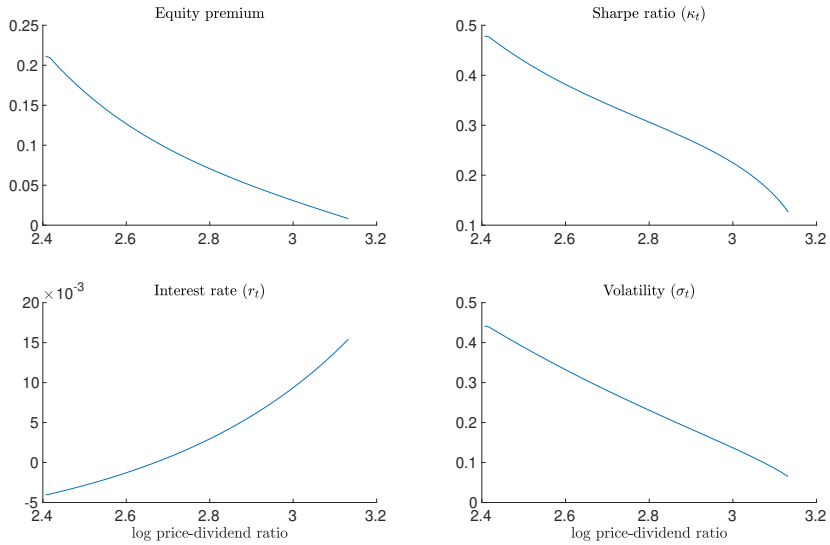

Figure 2: Calibration results. Equity premium, market price of risk (Sharpe ratio), interest rate, and stock return volatility as a function of the log price-dividend ratio of levered equity. The range of values of the log-price dividend ratio corresponds to the interval between the bottom and top 5-th percentiles of the stationary distribution of the log price-dividend ratio in the model. The equity premium and volatility are adjusted for leverage.

rate is re-assuring for an additional reason: Given equation (18), the interest rate is equal (up to an additive constant) to the consumption growth of existing agents. Accordingly, the low variation of the interest rate shows that the model does not require very strong variation in (the drift of) existing agents' consumption growth to deliver a realistic Sharpe ratio.

Finally, Table 2 reports results from standard predictability regressions in model simulations and in the data. As is well understood, the finite sample properties of predictability regressions make it very hard to precisely estimate regression coefficients over samples that are of length similar to the data. For this reason, we follow the standard practice of reporting both the mean and the 95% range of the model-implied regression estimates, which we obtain from simulating the model over 1000 independent 100-year-long samples. The table shows that the log-price-dividend ratio is a strong predictor of excess returns inside the model. The magnitudes for the R^2 of these predictability regressions are quite similar to the data, while the point estimates are somewhat larger in the model than in the data, but with the data estimate being well within the 95% range of the regression estimates obtained in simulations.

While throughout the paper we assume the difference in the depreciation rates η_t to be endoge-

Year	β (Data)	β (Model)	R^2(Data)	R^2(Model)
1	-0.130	-0.263	0.040	0.068
		[-0.562 0.020]		[0.002 0.162]
3	-0.350	-0.655	0.090	0.169
		[-1.260 0.029]		[0.005 0.377]
5	-0.600	-0.923	0.180	0.239
		[-1.691 -0.001]		[0.007 0.500]
7	-0.750	-1.114	0.230	0.289
		[-1.924 -0.043]		[0.009 0.564]

Table 2: Long-horizon regressions of excess returns on the log P/D ratio. The simulated data are based on 1000 independent simulations of 100-year long samples. For each of these 100-year long simulated samples, we run predictive regressions of the form $\log R^e_{t\to t+h} = \alpha + \beta \log(P_t/D_t)$, where $\log R^e_{t\to t+h}$ denotes the time-t gross excess return over the next h years. We report the mean values for the coefficient β and the R^2 of these regressions, along with the 95% confidence interval reported in square brackets below the estimate..

nous, the same results obtain if η_t is an exogenous process, specified to have the same dynamics as the endogenously derived $\eta(q_t)$. Given the theoretical focus of this paper and our goal to illustrate the interactions between the wealth distribution and discount rates, we chose to abstract from fundamental shocks, which would mostly be a distraction. However, the quantitative conclusion that the model can deliver volatile valuations without relying on dividend or interest rate variations is not dependent on the nature of shocks.

7 Conclusion

This paper focuses on the joint determination of discount rates and the distribution of wealth. The model features three key elements: a) there exist several assets, with different sensitivities to discount rates; b) agents have heterogeneous exposures to these assets; and c) only some agents are marginal for pricing assets at a given point in time. As a consequence, fluctuations in the components of wealth cause wealth redistribution amongst different groups of agents, which in turn feeds back into these fluctuations.

The starkest implication of this feedback loop is the possibility of self-fulfilling anticipations of future discount rates. Specifically, shifts in expectations of future discount rates can change the wealth composition, and consequently redistribute wealth, in such a way that the resulting

consumption-saving decisions confirm the anticipated discount rates.

As part of our analysis, we provide a new interpretation of the source of indeterminacy in models with rational bubbles. Contrary to the conventional wisdom, which considers the indeterminacy of equilibrium to be the direct consequence of the indeterminate magnitude of the bubble, we argue that the presence of a bubble is just an ancillary feature to trigger the interactions between the wealth distribution and the discount rates that we highlight in the context of our model.

Besides expanding the scope of the analysis to situations in which bubbles cannot exist, our model presents the advantage that the useful Campbell-Shiller-type decompositions of asset price fluctuations continue to apply. Utilizing these present value relations, one can confront the model with the data in the same way as any conventional asset pricing model. The feedback effects that we highlight can help reconcile volatile asset prices, high equity premiums, and predictable excess returns on the one hand with substantially less volatile real quantities (consumption, dividends, etc.) on the other.

Finally, while the paper maintains throughout the assumption that all uncertainty is extrinsic, we view this assumption primarily as a useful way to sharpen and clarify our results. The feedback effects that we highlight, along with their asset pricing implications, continue to obtain in versions of the model in which the redistributional shocks take an intrinsic form.

References

Abel, A. B. (1999). Risk premia and term premia in general equilibrium. *Journal of Monetary Economics 43*(1), 3–33.

Abel, A. B. (2003). The effects of a baby boom on stock prices and capital accumulation in the presence of social security. *Econometrica 71*(2), 551–578.

Abel, A. B., N. G. Mankiw, L. Summers, and R. Zeckhauser (1989). Assessing dynamic efficiency: Theory and evidence. *Review of Economic Studies 56*(1), 1–19.

Aghion, P. and P. Howitt (1992). A Model of Growth through Creative Destruction. *Econometrica 60*(2), 323–351.

Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance 59*(4), 1481–1509.

Barlevy, G. and P. Veronesi (2000). Information acquisition in financial markets. *Review of Economic Studies 67*(1), 79–90.

Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics 121*(3), 823–66.

Başak, S. (2000). A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk. *Journal of Economic Dynamics and Control 24*(1), 63–95.

Başak, S., D. Cass, J. M. Licari, and A. Pavlova (2008). Multiplicity in general financial equilibrium with portfolio constraints. *Journal of Economic Theory 142*(1), 100–127.

Benhabib, J. and R. Farmer (1999). Indeterminacy and sunspots in macroeconomics. In J. B. Taylor and M. Woodford (Eds.), *Handbook of Macroeconomics*, Volume 1, Part A, Chapter 06, pp. 387–448.

Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy 93*(2), 223–247.

Blanchard, O. J. and M. W. Watson (1982, July). Bubbles, rational expectations and financial markets. NBER Working Papers 0945, National Bureau of Economic Research, Inc.

Brunnermeier, M. (2008). Bubbles. In L. Blume and S. Durlauf (Eds.), *New Palgrave Dictionary of Economics* (second ed.).

Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy 107*(2), 205–251.

Cass, D. and K. Shell (1983). Do sunspots matter? *Journal of Political Economy 91*(2), 193–227.

Constantinides, G., J. B. Donaldson, and R. Mehra (2002). Junior can't borrow: A new perspective on the equity premium puzzle. *The Quarterly Journal of Economics 117*(1), 269–296.

DeMarzo, P., R. Kaniel, and I. Kremer (2008). Relative wealth concerns and financial bubbles. *Review of Financial Studies 21*(1), 19–50.

Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica 60*(2), 353–394.

Farmer, R. E. (2018). Pricing assets in a perpetual youth model. *Review of Economic Dynamics 30*, 106–124.

Gârleanu, N. and S. Panageas (2015). Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. *Journal of Political Economy 123*, 670–685.

Gârleanu, N. and S. Panageas (2019). Heterogeneity and asset prices: A different approach. Unpublished working paper, UC Berkeley and UCLA.

Gârleanu, N., S. Panageas, and J. Yu (2015). Financial entanglement: A theory of incomplete integration, leverage, crashes, and contagion. *American Economic Review 105*(7), 1979–2010.

Giglio, S., M. Maggiori, and J. Stroebel (2016). No-Bubble Condition: Model-Free Tests in Housing Markets. *Econometrica 84*, 1047–1091.

Gomes, F. and A. Michaelides (2005). Optimal life-cycle asset allocation: Understanding the empirical evidence. *Journal of Finance 60*(2), 869–904.

Gomez, M. (2017). Asset prices and wealth inequality. Working Paper, Columbia University.

Harrison, J. M. and D. M. Kreps (1978). Speculative investor behavior in a stock market with heterogeneous expectations. *Quarterly Journal of Economics 92*, 323–336.

Karlin, S. and H. Taylor (1981). *A Second Course in Stochastic Processes*. Academic Press.

Maurer, T. (2017). Asset pricing implications of demographic change. Working Paper, Olin School of Business.

Miao, J. and P. Wang (2018). Asset bubbles and credit constraints. *American Economic Review 108*(9), 2590–2628.

Romer, P. M. (1990, October). Endogenous technological change. *Journal of Political Economy 98*(5), 71–102.

Santos, M. S. and M. Woodford (1997). Rational asset pricing bubbles. *Econometrica 65*(1), 19–58.

Scheinkman, J. and W. Xiong (2001). Overconfidence, short-sale constraints, and bubbles. Unpublished working paper. Princeton University.

Schneider, A. (2017, November). Risk sharing and the term structure of interest rates. Unpublished working paper, Federal Reserve Board of Governors.

Schroder, M. and C. Skiadas (1999). Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory 89*, 68–126.

Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica 53*(6), 1499–1528.

Zentefis, A. (2019). Self-fulfilling asset prices. Working paper, Yale SOM.

Appendix

A Proofs

Proof of Lemma 1. The absence of bubbles together with the assumption of a unit elasticity of substitution implies that aggregate consumption is given by $C_t = \beta\left(\bar{W}_t + \bar{H}_t\right)$, where $\bar{W}_t$ is the present value of dividends of all existing firms

$$\bar{W}_t = \int_{-\infty}^{t} q_{t,s}^d \; D_{t,s} ds = q_t^d \int_{-\infty}^{t} D_{t,s} ds = \alpha q_t^d Y_t, \tag{40}$$

and H_t is the present value of the earnings of all existing workers.

$$\bar{H}_t = \int_{-\infty}^{t} q_{t,s}^l w_{t,s} l_{t,s} ds = (1-\alpha)\, q_t^l Y_t. \tag{41}$$

Combining goods market clearing ($C_t = Y_t$) with (40) and (41), and re-arranging leads to (15). ■

Proof of Lemma 2. The present value of all newly-born workers' wages is given by $(1-\alpha)\,(g+\delta^l) q_t^l Y_t$, while the respective value of all newly created firms is $\alpha(g+\delta_t^d) q_t^d Y_t$. Imposing the intertermporal budget constraint and noting that the consumption-to-wealth ratio for investors with unit elasticity of substitution is β implies that per-capita consumption of the newly born is given by (21). ■

Proof of Lemma 3. Since the value function of a newly-born person is logarithmic in wealth, an entrepreneur indexed by $i \in [0, \bar{\varepsilon}]$ prefers the risky to the riskless choice if and only if

$$\pi \log\left(\xi\,(i)\, q_t^d \alpha Y_t\right) + (1-\pi) \log\left(q_t^l \frac{g+\delta^l}{1-\bar{\varepsilon}+(1-\pi)\,i}\,(1-\alpha)\, Y_t\right) \geq \log\left(\psi q_t^d \alpha Y_t\right). \tag{42}$$

The left hand side gives the value function of trying the risky choice which succeeds with probability π and fails with probability $(1-\pi)$, in which case the entrepreneur shares the labor income accruing to her cohort. The right hand side is the (certain) payoff of the riskless choice. Simplifying and

re-arranging gives

$$(1-\pi)\log\left(\frac{q_t^d}{q_t^l}\right) \leq \pi\log\left(\xi\left(i\right)\alpha\right) + (1-\pi)\log\left(\frac{\left(g+\delta^l\right)\left(1-\alpha\right)}{1-\bar{\varepsilon}+\left(1-\pi\right)i}\right) - \log\left(\psi\alpha\right). \tag{43}$$

We note that the right-hand side of this inequality is a decreasing function of i, so that the set of entrepreneurs making the risky choice takes the form $[0, \zeta_t]$, as stated in the text. It further follows that decreasing $\frac{q_t^d}{q_t^l}$ — which by Lemma 1 is equivalent to decreasing q_t^d — thus the left-hand side of (43), results in a weakly larger cutoff ζ_t. (Whenever (43) holds with equality for $i = \zeta_t$, the monotonicity of ζ_t in q_t^d is strict.) ■

Proof of Proposition 1. Let $\tilde{A}\left(q;\eta\right) \equiv \left(\beta+\eta\right)q - \beta\alpha\eta q^2 - 1$ and define the function

$$\eta^*(q) = \frac{1-\beta q}{q(1-\alpha\beta q)} \tag{44}$$

for $q \in (0, \frac{1}{\alpha\beta})$. By construction, $\tilde{A}(q;\eta^*(q)) = 0$. It is easy to verify (e.g., by direct differentiation) that η^* decreases strictly in q. Note also that $\frac{\partial}{\partial\eta}\tilde{A}(q;\eta) = q(1-\alpha\beta q) > 0$.

Let $\eta(q)$ be continuously differentiable and decreasing with the following properties: (i) $\eta(q_i) = \eta^*(q_i)$ for $i \in \{1,2,3\}$; (ii) $\eta'(q_i) > \eta^{*\prime}(q_i)$, $i \in \{1,3\}$; (iii) $\eta'(q_2) < \eta^{*\prime}(q_2)$. Given that $\frac{\partial}{\partial\eta}\tilde{A} > 0$, these properties ensure that the proposition holds. Specifically,

$$\frac{d\tilde{A}}{dq}(q,\eta(q)) = \frac{\partial\tilde{A}}{\partial q} + \frac{\partial\tilde{A}}{\partial\eta}\frac{d\eta}{dq} \tag{45}$$

for any $\eta(q)$, and $\frac{d\tilde{A}}{dq}(q,\eta^*(q)) = 0$ since $\tilde{A}(q;\eta^*(q))$ is constant by the definition of η^*. Consequently, $\frac{d\tilde{A}}{dq}|_{q_i}(q,\eta(q))$ is strictly positive for $i \in \{1,3\}$ and strictly negative for $i = 2$.[16]

It remains to show that, for such a function η as chosen above, model primitives exist under which $\eta(q)$ is an equilibrium outcome. In particular, we need to show the existence of an appropriate — exogenous — function ξ, based on which the cutoff $\zeta(q)$ is determined. These two functions obey two restrictions: equations (43) — as an equality — and (7).

With $i = \zeta(q)$, equation (43) provides the function ξ on the domain constituted by the image

[16]Note that properties (ii) and (iii) require that η be flatter than η^* around the extreme q_i, and steeper around q_2. This is the precise sense in which η must be sufficiently steep to ensure that q_2 is a stable equilibrium.

of ζ. Rewriting equation (7) as

$$\eta(q) = \pi \int_0^{\zeta(q)} \xi(i)di + (\bar{\varepsilon} - \zeta(q))\,\psi - (\delta^l + g), \tag{46}$$

we have

$$\eta'(q) = (\pi\xi(\zeta(q)) - \psi)\,\zeta'(q), \tag{47}$$

or

$$\zeta'(q) = \frac{\eta'(q)}{\pi\xi(\zeta(q)) - \psi}. \tag{48}$$

Given $\xi(\zeta(q))$ from (43), this is a first-order ODE in $\zeta(q)$.

We wish that a decreasing solution to this ODE exist on $[q_1, q_3]$ with image in $[0, \bar{\varepsilon}]$. To ensure the existence of such a solution, we can build one as follows under appropriate parameter choices. Specifically, we'll find appropriate values for $\delta^l + g$, $\bar{\varepsilon}$, and ψ, having fixed the other parameters arbitrarily, that result in functions $\xi(\cdot)$ and $\zeta(\cdot)$ having all the desired properties.

We start by letting $\zeta(q_3) = 0$, so that from (46) we have

$$\delta^l + g = \bar{\varepsilon}\psi + \eta(q_3). \tag{49}$$

By considering ψ increasing without bounds subject to (49) and a fixed $\bar{\varepsilon}$, we have

$$\frac{\psi}{\delta^l + g} \to \frac{1}{\bar{\varepsilon}} \tag{50}$$

as $\psi \to \infty$, which we use in conjunction with equation (43) to derive the existence of constants c_i with $0 < c_1 < c_2 < \infty$ such that

$$\lim_{\psi\to\infty} \frac{\xi(\zeta(q))}{\psi} \in (c_1, c_2) \times \bar{\varepsilon}^{\frac{\pi-1}{\pi}}. \tag{51}$$

These constants can be chosen independently of $q \in [q_1, q_3]$, $\zeta \in [0, \bar{\varepsilon}]$, and $\bar{\varepsilon} \in [0, \tilde{\varepsilon}]$ for a fixed, arbitrary, $\tilde{\varepsilon} < 1$. All other parameters are fixed. It follows that $\bar{\varepsilon}$ can be chosen small enough to imply $\xi(\zeta(q))/\psi > 2/\pi$, say; then, $\pi\xi(\zeta(q)) - \psi > \psi \to \infty$ as $\psi \to \infty$.

With $\bar{\varepsilon}$ fixed, we now let ψ sufficiently large so that $\pi\xi(\zeta(q)) - \psi > M$ for a constant $M > \max_q |\eta'(q)|(q_3 - q_1)/\bar{\varepsilon}$. We consequently have that the solution to the ODE (48) has the property

$$\zeta(q_1) = \int_{q_1}^{q_3} \frac{-\eta'(q)}{\pi\xi(\zeta(q)) - \psi} dq \tag{52}$$

$$< \max_q |\eta'(q)| \frac{q_3 - q_1}{M} < \bar{\varepsilon}. \tag{53}$$

In summary, we can express ξ as a function of q and $\zeta(q)$ from the indifference condition of the entrepreneur $\zeta(q)$. Then the equation relating the displacement quantity δ^d to the summation of $\xi(i)$ until $\zeta(q)$ provides an ODE in ζ. Finally, to ensure that the solution to the ODE falls in the interval $[0, \bar{\varepsilon}]$, it suffices to choose $\bar{\varepsilon}$ small enough and then let ψ become large, with $\delta^l + g$ ensuring $\zeta(q_3) = 0$. ■

Proof of Proposition 2. We start with the observation that goods market clearing implies that in any equilibrium (stochastic or deterministic) the consumption of existing investors — who are marginal for pricing assets — must be locally deterministic,[17] which in turn implies that this economy cannot have a risk premium for any asset. Additionally, equations (18) and (20) continue to hold even if q_t^d is stochastic, as do Lemmas 1, 2, and 3. Accordingly q_t^l and r_t continue to be given by (15) and (22) respectively. The main equation that needs to be modified is equation (17). By the definition of q_t^d we have that

$$e^{-\int_s^t r_v dv} q_{t,s}^d D_{t,s} + \int_s^t e^{-\int_s^u r_v dv} D_{u,s} du \equiv E_t \int_s^\infty e^{-\int_s^u r_v dv} D_{u,s} du. \tag{54}$$

Applying Ito's Lemma to both sides of (54), substituting r_t from (22), recognizing that the right hand side of (54) is a martingale implies that the drift of q_t^d in any stochastic equilibrium must necessarily be given by $A\left(q_t^d\right)$. Moreover, if the dynamics of q_t^d are given by (27), the Feynman-Kac formula implies that q_t^d satisfies (13). In a nutshell, if agents perceive that the dynamics of asset prices are given by (27), and the interest rate is given by (22), then the resulting optimal dynamics of consumption will be such that the market-clearing interest rate will be given by (22) and the equilibrium (arbitrage-free) price of each firm will indeed be given by $q_t^d D_{t,s}$.

[17] This follows because aggregate consumption is deterministic, while newly born agents arrive and consume at a positive, finite rate.

To prove that q_t^d is stationary, we start by defining

$$s(q) \equiv \exp\left\{-\int^q \frac{2A(\xi)}{\sigma^2(\xi)} d\xi\right\} \tag{55}$$

and we note that an implication of Assumption (26) is that for $q \in (q^{\max} - \varepsilon, q^{\max})$ and ε small enough there exists a finite $\nu > 1$ such that

$$\frac{s(q)}{s(q^{\max} - \varepsilon)} = \exp\left\{-\int_{q^{\max}-\varepsilon}^q \frac{2A(\xi)}{\sigma^2(\xi)} d\xi\right\} > \exp\left\{-\int_{q^{\max}-\varepsilon}^q \frac{\nu}{q^{\max} - \xi} d\xi\right\} = \left(\frac{q^{\max} - q}{\varepsilon}\right)^{-\nu}.$$

We infer that the speed measure $S(q) = \int^q s(\eta)\, d\eta)$ satisfies $S(q, q^{max}) = \infty$, in the limit sense of Karlin and Taylor (1981) (equation (6.7) on page 227). Consequently, q^{max} is not an attainable boundary. The calculations of Karlin and Taylor (1981), Example 8 on page 239, can similarly be adapted to show that q^{max} is in fact an entrance boundary. (A similar argument applies to the boundary $q = q^{\min}$.) Then, as Karlin and Taylor (1981) shows, a stationary distribution exists. ■

Proof of Proposition 3. (i) Note first that, in this case, $q^{**} < q^*$. If $q_t < q^{**}$ for some t, then $\dot{q}_t < 0$, and since $\dot{q}$ given by equation (30) is concave in q, it follows that $q_s \to -\infty$ as $s \to \infty$, which is absurd.

If, on the other hand, $q_t > q^{**}$ for some q, then equation (30) implies that q_t converges to the steady state value of q^*. A value q^* is inconsistent with equilibrium, because it implies a negative bubble value:

$$\frac{b_t}{Y_t} = \frac{P_t}{Y_t} - \alpha q^* = \frac{1}{\beta} - q^* = \frac{1}{\beta} - \frac{1}{g + \delta} < 0. \tag{56}$$

The value of the stock market, however, can only be lower than the present value of dividends if, at some point in the future, it becomes negative with positive probability, which is incompatible with free disposal.

(ii) If $\beta < g + \delta$, then the stable solution to equation (30) is a steady-state equilibrium, as is the unstable solution $q_t = q^*$. In the former case, the bubble value is strictly positive. The stability of the steady-state equilibrium $q_t = q^{**}$ allows for the existence of a continuum of equilibrium paths, all of which converging to q^{**}. Any such path must start from $q_0 \in (q^*, q^{**}]$.

Differentiating equation (32) gives

$$\begin{aligned}\dot{b}_t &= \frac{1}{\beta}\dot{Y}_t - \dot{q}_t Y_t - q_t \dot{Y}_t = Y_t\left(g\left(\frac{1}{\beta} - q_t\right) - \left(r_t + \delta_t - \frac{1}{q_t}\right) q_t\right) \\ &= Y_t\left(\frac{g}{\beta} - (r_t + g + \delta)\, q_t + 1\right) \\ &= Y_t\left(\frac{r_t}{\beta} - r_t q_t\right) + Y_t\left(\frac{g - r_t}{\beta} + 1 - (g+\delta) q_t\right) \\ &= r_t b_t, \end{aligned} \tag{57}$$

where the last equality is obtained by plugging in

$$\frac{r_t}{\beta} = \frac{g}{\beta} + 1 - (g+\delta) q_t, \tag{58}$$

which follows from from equations (28) and (29), and using the definition of b_t.

The last statement of the proposition follows from (58) and $1 - (g+\delta)q_t > 1 - (g+\delta)q_0 > 0$. ■

Proof of Proposition 4. In a stochastic economy the SDF in the presence of annuities is given by

$$d\log m_t = -\left(r_t + \lambda\right) dt - \frac{\kappa_t^2}{2} dt - \kappa_t dB_t, \tag{59}$$

where κ_t is the Sharpe ratio. In the special case in which preferences are given by (33), the SDF satisfies the evolution equation

$$d\log(m_t) = f_V(c, V) dt + d\log(f_c(c, V), \tag{60}$$

which results in

$$d\log m_t = \gamma\left(\log c_{t,s} - \beta U_{t,s}\right) dt - \beta dt + \gamma dU_{t,s} - d\log c_{t,s}. \tag{61}$$

As is shown in Garleanu and Panageas (2015), the fact that the investment opportunity set (r_t, κ_t), the hazard rate of death, and the discount rate are the same for all investors implies that $d\log c_{t,s}$ is independent of s for $s < t$. Accordingly, $dU_{t,s}$ is independent of s and we omit the

subscript s in equation (61). For any $s < t$, the definition of U_t implies

$$e^{-\beta t}U_t + \int_s^t e^{-\beta u}\left(\log(c_u) + \frac{\gamma}{2}d[U]_u\right) = E_t\int_s^\infty e^{-\beta u}\left(\log(c_u)\,du + \frac{\gamma}{2}d[U]_u\right). \tag{62}$$

Differentiating both sides with respect to t, noting that the right hand side is a martingale, and using the martingale representation theorem implies that

$$-\beta U_t dt + dU_t + \left(\log(c_t) + \frac{\gamma}{2}\sigma_{U,t}^2\right)dt = \sigma_{U,t}dB_t.$$

Upon re-arranging we obtain

$$\gamma dU_t = \left[-\gamma\log(c_t) - \frac{\gamma^2}{2}\sigma_{U,t}^2 + \gamma\beta U_t\right]dt + \gamma\sigma_{U,t}dB_t \tag{63}$$

Plugging (63) into (61) and re-arranging gives

$$d\log m_t = -\beta dt - d\log c_t - \frac{\gamma^2}{2}\sigma_{U,t}^2 dt + \gamma\sigma_{U,t}dB_t. \tag{64}$$

Market clearing requires that $d\log c_t$ is locally deterministic, since aggregate consumption $\lambda\int_{-\infty}^t e^{-\lambda(t-s)}c_{t,s}ds$ is deterministic. We can therefore write $d\log c_t = \frac{\dot{c}_t}{c_t}$. Comparing (59) with (64) and matching drift and diffusion terms results in

$$\frac{\dot{c}_t}{c_t} = -(\rho - r_t), \tag{65}$$

which is equation (18), and $\kappa_t = -\gamma\sigma_{U,t}$.

Since equation (18) holds, the interest rate continues to be given by (22) and Lemma 1 continues to hold. Accordingly, $r_t = r\left(q_t^d\right)$, that is, the interest rate is a function of q_t^d. We conjecture (and verify shortly) that the Sharpe ratio is also a function of q_t^d, so that we can write $\kappa_t = \kappa\left(q_t^d\right)$. Then the requirement that the SDF-discounted gains process from investing in a stock be a martingale leads to (36).

To verify that the Sharpe ratio is a function of q_t^d, we now compute the dynamics of U_t. Expressing equation (18) as $d\log c_t = -(\rho - r_t)\,dt$, we obtain

$$U_t = E_t\int_t^\infty e^{-\beta(u-t)}\log(c_u)\,du + \frac{\gamma}{2}E_t\int_t^\infty e^{-\beta(u-t)}d[U]_u.$$

We next write

$$E_t \int_t^\infty e^{-\beta(u-t)} \log(c_u)\, du = E_t \int_t^\infty e^{-\beta(u-t)} \left(\log(c_t) + \left(\int_t^u (r_x - \rho)\, dx \right) \right) du$$
$$= \frac{\log c_t}{\beta} + E_t \int_t^\infty e^{-\beta(u-t)} \int_t^u (r_s - \rho) ds\, du.$$

Since $r_t = r(q_t)$ and q_t is Markovian (under the assumption $\kappa_t = \kappa(q_t)$), it follows that

$$U_t = \frac{\log(c_t)}{\beta} + \tilde{Z}(q_t) \tag{66}$$

for an appropriate function $\tilde{Z}$. Since c_t has zero volatility, we have

$$\sigma_{U,t} = \tilde{Z}'(q_t)\sigma(q_t), \tag{67}$$

confirming the conjecture that κ_t is a function of q_t.

Finally, plugging (66) on the left-hand side of equation (62) and computing the drift of this term we obtain the ODE

$$\frac{\sigma^2}{2}\tilde{Z}''(q) + \left(A(q) + \kappa(q)\sigma(q)\right)\tilde{Z}'(q) - \beta\tilde{Z}(q) + \frac{r(q) - \rho}{\beta} + \frac{\gamma}{2}\left(\tilde{Z}'(q)\right)^2 \sigma(q)^2 = 0. \tag{68}$$

Equation (22) and Lemma 2 imply $\widetilde{Z}(q_t) = Z(q_t) + const.$, where $Z(q_t)$ solves (38).

To summarize, the construction of equilibrium starts by making a choice of $\sigma(q_t)$ subject to the same technical conditions as in Proposition 2. Conditional on this choice, and given the function $\eta(q)$, we obtain a solution to (38). Then the equilibrium dynamics of q_t are given by the stochastic differential equation (36) and the dynamics of the interest rate by (22). ■

B Bubbles introduced by arriving cohorts

We revisit here the model of Section 4 and allow the incoming cohorts to introduce bubbles. We show that the results of Section 4 continue to hold. Namely, as long as $\beta < g + \delta$, we construct equilibria in which arriving assets are priced above the fundamental value. In the interest of simplicity, we focus on the case in which the total value of the bubbles introduced by arriving

cohorts per unit of time is $\bar{b}Y_t$ for a given $\bar{b} > 0$ that is restricted appropriately in Lemma 4 below. Under this assumption, equation (29) becomes

$$\frac{c_{t,t}}{Y_t} = \frac{\beta}{\lambda}\left((g+\delta)q_t + \bar{b}\right). \tag{69}$$

Accordingly, equation (30) becomes

$$\dot{q}_t = \left(\beta + g + \delta - \beta\bar{b}\right) q_t - (g+\delta)\beta q_t^2 - 1. \tag{70}$$

Lemma 4 *For any $\bar{b} < \frac{\beta+g+\delta-\sqrt{4(g+\delta)\beta}}{\beta}$ there exist two steady state values of q_t, lying in the interval $\left(\frac{1}{g+\delta}, \frac{1}{\beta}\right)$. In either of these equilibria the total detrended value of the bubble in the economy is positive in the long run.*

Proof of Lemma 4. Setting $\dot{q}_t = 0$ in equation (70) generates a quadratic equation for the steady-state values of q_t. This equation admits two real solutions as long as $\bar{b} < \frac{\beta+g+\delta-\sqrt{4(g+\delta)\beta}}{\beta}$. We omit the rest of the proof. ■

A practical consequence of Lemma 4 is that the aggregate value of the bubble does not disappear asymptotically in either steady state. If $\hat{q}^* < \hat{q}^{**}$ are the two stead-state values for q, then any initial value q_0 in the range $(\hat{q}^*, \frac{1}{\beta}]$ leads to the same steady state value $\hat{q}^{**}$ and the same value of the aggregate stock market $P_t = \left(\frac{1}{\beta} - (1-\alpha)\,\hat{q}^{**}\right) Y_t$, asymptotically. As a fraction of output, the total value of bubbles in the economy approaches $\frac{1}{\beta} - \hat{q}^{**} > 0$.

C General IES

Here we repeat the analysis of Section 3, but assuming that the per-period utility function is $U(c) = \frac{c^{1-\phi^{-1}}}{1-\phi^{-1}}$. Equation (18) becomes

$$\frac{\dot{c}_{t,s}}{c_{t,s}} = -\phi\left(\rho - r_t\right) \tag{71}$$

so that (19) now leads to

$$r_t = \beta + \phi^{-1}\left(g - \lambda\frac{c_{t,t}}{Y_t}\right). \tag{72}$$

To determine $\frac{c_{t,t}}{Y_t}$, we define f_t as the ratio of the present discounted value of an agent's future consumption to her current consumption:

$$f_t := \int_t^\infty e^{-\int_t^u (r_v+\lambda)dv} \left(\frac{c_{u,s}}{c_{t,s}}\right) du = \int_t^\infty e^{(\phi-1)\int_t^u r_v dv - (\rho\phi+\lambda)(u-t)} du, \tag{73}$$

where we used (71). Recognizing that the present value of consumption equals an agent's total wealth, we obtain an analog to Lemma 2,

$$\frac{c_{t,t}}{Y_t} = \frac{1}{f_t\lambda}\left((1-\alpha)(g+\delta^l)q_t^l + \alpha(g+\delta_t^d)q_t^d\right), \tag{74}$$

and an analog to Lemma 1:

$$(1-\alpha)\, q_t^l = f_t - \alpha q_t^d. \tag{75}$$

Substituting (75) into (74) and then into (72) leads to

$$r_t = \beta + \phi^{-1}\left(\delta^l - \alpha\eta_t \frac{q_t^d}{f_t}\right). \tag{76}$$

Finally, observing that δ_t^d continues to be exclusively a function of $\frac{q_t^d}{q_t^l}$ according to an equation analogous to (43), we conclude that the dynamics of the economy can be described in terms of the dynamics of $(q_t^d, f_t) =: (q_t, f_t)$. Differentiating (73), we have

$$\dot{q}_t = \left(r_t + \delta_t^d\right) q_t - 1 \tag{77}$$

$$\dot{f}_t = (\rho\phi + \lambda + (1-\phi)\, r_t)\, f_t - 1, \tag{78}$$

keeping in mind that r_t is given by (76).

We next study the dynamics of the system (77)–(78) starting with the special case corresponding to the limit $\phi \to 1$, i.e., $u(c) = \log(c)$. As shown by Proposition 1, we have then — under appropriate parameter choices — that the dynamical system (77)–(78) exhibits two unstable steady states and one saddle-path steady state whose stable arm involves $f_t = \rho + \lambda = \beta$ for all t and dynamics for q_t given by (25).

For $\phi \neq 1$ we have the following result.

Proposition 5 *Choose a set of parameters under which the conclusion of Proposition 1 obtains.*

For ϕ sufficiently close to one, the dynamical system (77)–(78) has a saddle-path-stable steady state.

Proof of Proposition 5. By assumption, when $\phi = 1$ there exists a (saddle-path) stable steady state with long run values for q_t and f_t given by q_2, respectively $\frac{1}{\beta}$. The continuity of the right-hand side of (77)–(78) implies that for $\varepsilon := |\phi - 1| > 0$ sufficiently small, there exists a steady state $\left(q^{SS}, f^{SS}\right)$ in the neighborhood of $\left(q_2, \frac{1}{\beta}\right)$. At this steady state, the Jacobean matrix is

$$J = \begin{bmatrix} \left(\frac{\partial r}{\partial q} + \frac{\partial \delta^d}{\partial q}\right) q^{SS} + \frac{1}{q^{SS}} & \left(\frac{\partial r}{\partial f} + \frac{\partial \delta^d}{\partial f}\right) q^{SS} \\ -(\phi - 1) f^{SS} \frac{\partial r}{\partial q} & \frac{1}{f^{SS}} - (\phi - 1) f^{SS} \frac{\partial r}{\partial f} \end{bmatrix}. \tag{79}$$

As $\varepsilon \to 0$, $(q^{SS}, f^{SS}) \to (q_2, \frac{1}{\beta})$ and the eigenvalues of J converge to $(\frac{dr}{dq} + \frac{d\delta^d}{dq})q^{(2)} + \frac{1}{q^{(2)}} < 0$ and $\beta > 0$, where the negativity of the first eigenvalue follows from $A'(q_2) < 0$ (Proposition 1). The continuity of J in (q, f) implies that the system remains saddle-path stable for ϕ sufficiently close to unity. ■

Unsurprisingly, when the IES is close to one, the model behaves similarly to our benchmark (log) model, and in particular shares its properties. For qualitative departures from the benchmark model, larger parameter deviations are necessary. One such departure that is interesting to establish is that new-firm entry can be positively related to the valuation ratio $q = q^d$, which is the opposite of the benchmark model.

The main idea behind a result of this nature is to recognize that, in order to pair higher entry, thus depreciation, rates with the higher asset valuation ratios, the interest rates must react in the opposite direction, and even more than the depreciation rates, to changes in the state variable. Thus, when depreciation rates are high, so that marginal agents' consumption growth is low, the interest rates have to be very low, which follows if the IES is sufficiently smaller than one.

Here is a formal result.

Proposition 6 *For ϕ close enough to zero, a specification of $\delta_t^d = \delta^d(q_t/f_t)$ exists so that the dynamical system (77)–(78) has a stable steady state and δ_t^d and q_t are positively correlated.*

Proof of Proposition 6. Let $z_t := \frac{q_t}{f_t}$, and

$$\delta(z) = \delta^l + \frac{\delta^l}{\alpha} z^{-1} + \phi\,(\alpha z)^{-1} \left((\beta - a) + \left(\delta^l + \frac{\delta^l}{\alpha} z^{-1} \right) - g(z - z^{SS}) \right), \tag{80}$$

so that, from (76), we have

$$r(z) = a - \left(\delta^l + \frac{\delta^l}{\alpha} z^{-1}\right) + g(z - z^{SS}). \tag{81}$$

Here, a is a constant, which we can specify later to help achieve our objective. Similarly for the increasing function g, with $g(0) = 0$; the constant z^{SS} will be chosen to ensure that it equals the steady-state value of z.

We need to be able to ensure two properties. First, that the system admits a stable steady-state solution; and second, that, in a neighborhood of such a steady state, q increases on a path on which z decreases. Our strategy is as follows: specify z^{SS} and a so that r and $r + \delta$ increase in z around the steady state, which equals z^{SS}, and the steady state is stable. Under these properties, it follows immediately that, for low — and therefore increasing — z, thus low discount rates, both q and f decrease.

To ensure stability, we start by computing the determinant of J, noting that $\frac{\partial r}{\partial q} q = r'z$, $\frac{\partial r}{\partial q} f = r'$, $\frac{\partial r}{\partial f} q = -r'z^2$, and $\frac{\partial r}{\partial f} f = -r'z$, as well as analogous relations for the derivatives of δ^d. Multiplying this determinant by q^2/z, we obtain

$$z(r + \delta^d)' + \left(1 - (1 - \phi)r'\right), \tag{82}$$

which we want to be negative at the steady state. As for the steady state, from (77)–(78) we have the equation

$$\left(r\left(z^{SS}\right) + \delta^d\left(z^{SS}\right)\right) z^{SS} = \rho\phi + \lambda + (1 - \phi) r\left(z^{SS}\right). \tag{83}$$

Plugging the resulting expression for z^{SS} into equation (82), we obtain the condition

$$\frac{(r + \delta)'}{r + \delta} < \frac{(1 - \phi)r' - 1}{(1 - \phi)r + \rho\phi + \lambda}, \tag{84}$$

when evaluated at $z = z^{SS}$.

We now observe that, if we choose a value $z^{SS} < \min\left\{1, \sqrt{\alpha^{-1}\delta^l}\right\}$ and solve for a such that

the limit of equation (83) as $\phi \to 0$, i.e.,

$$z^{SS} = \frac{\lambda + a - \delta^l - \alpha^{-1}\delta^l (z^{SS})^{-1}}{a}, \tag{85}$$

holds,[18] then the corresponding limit of condition (84) becomes

$$\frac{g'(0)}{a} < \frac{\alpha^{-1}\delta^l \left(z^{SS}\right)^{-2} + g'(0) - 1}{r\left(z^{SS}\right) + \lambda}. \tag{86}$$

This inequality is clearly satisfied for $g'(0) > 0$ small enough, given that our initial choice of z^{SS} satisfies $\alpha^{-1}\delta^l \left(z^{SS}\right)^{-2} - 1 > 0$.

Since all the terms in (83) and (84) are continuous in ϕ^{-1} — and the limits as $\phi^{-1} \to 0$ exist and are finite, as we saw above — the conclusion holds for non-zero values of ϕ^{-1}, as well. Namely, a stable steady-state equilibrium exists characterized by $r'\left(z^{SS}\right) + (\delta^d)'\left(z^{SS}\right) > 0$ and $r'\left(z^{SS}\right) > 0$. It then follows, as we argued above, that, in a neighborhood of this steady-state equilibrium, high entry δ^d is accompanied by high q — as well as high f and low $\frac{q}{f}$. ■

D Extensions

D.1 Firm creation and growth

One of the assumptions of our analysis is that the arrival of new firms only disrupts existing ones, rather than also leading to extra growth. It is straightforward to relax this assumption without affecting the key insights of the analysis. We provide a sketch of the argument here.

We start by allowing aggregate growth to be time varying, by letting $g_t \equiv \frac{\dot{Y}_t}{Y_t}$, and generalizing equation (2) to

$$w_{t,s} \equiv \frac{(1-\alpha)\left(\delta^l + g_s\right) Y_t e^{-\delta^l (t-s) - \int_s^t g_u du}}{l_{t,s}}. \tag{87}$$

With this specification it is straightforward to check that $\int_{-\infty}^{t} w_{t,s} l_{t,s} ds = (1-\alpha) Y_t$. We continue to assume that dividends are given by (4). The main substantive departure from Section 2.5 is that

[18] This entails only solving a linear equation in a, which has a solution for any $z^{SS} \in (0,1)$.

we drop Assumption 1. Instead, we let

$$\nu_t \equiv \pi \int_0^{\zeta_t} \xi^i di + (\bar{\varepsilon} - \zeta_t)\, \psi$$

and assume that the depreciation of existing firms' profits is given by $\delta_t^d = \chi \nu_t - g$ for strictly positive constants $\chi \leq 1$ and g. Note that in the special case $\chi = 1$ we recover the setup of Section 2.5. With this specification, aggregate dividends grow at the rate

$$\frac{\dot{D}_t^A}{D_t^A} = \underbrace{\frac{\int_{-\infty}^t \dot{D}_{t,s} ds}{D_t^A}}_{=-\delta_t^d} + \underbrace{\frac{D_{t,t}}{D_t^A}}_{=\nu_t} = g + (1-\chi)\, \nu_t. \tag{88}$$

Repeating the analysis of Section 3, we obtain that the growth rate of the economy is given by

$$\frac{\dot{Y}_t}{Y_t} = g_t = g + (1-\chi)\, \nu_t,$$

and aggregate dividends are a constant fraction α of aggregate output: $D_t^A = \alpha Y_t$.

High values of χ imply that increased new firm creation impacts mostly the profits of existing firms without affecting aggregate growth substantially, while low values of χ imply that new firm creation adds to aggregate growth without substantially affecting the profits of existing firms.

Extending the remainder of the analysis is straightforward. Performing the same calculations as in the text we conclude that equation (25) continues to hold, with $\eta(q_t) = \chi \nu(q_t) - g + \delta^l$. As long as χ is different from zero, all the qualitative conclusions of the paper pertaining to indeterminacy, multiple equilibria, etc., remain intact.

D.2 Different depreciation rates within income groups

In the baseline version of the model we assumed that there are two types of income processes, namely "dividends" and "earnings." Fundamentally, these two income streams are constructed so that a) once aggregated across agents they amount to constant fractions of aggregate income, b) they have different depreciation processes (δ^l, δ_t^d), and b) their difference (η_t) is endogenous and dependent on the arrival rate of new entities that produce the income stream $D_{t,s}$. It is immaterial for our results how the present value of these two income processes is allocated within the arriving cohort.

For instance, assume that workers are of two kinds, "high skilled" and "low skilled." The endowment process of the high-skilled workers is given by (2), except that α now should be interpreted as the share of aggregate output accruing to "high-skilled" labor. The endowment of low-skilled workers born at time s is a constant fraction of $D_{t,s}$, with the remaining fraction paid out as profits to owners of the firms created at time s.[19] If entrepreneurs fail, they obtain the income process of a high skilled worker. Moreover, the value of the firms they create reflects that now their firms obtain a fraction of $D_{t,s}$.

This version of the model is equivalent to the baseline model (with a modified parameter α). The reason is that markets are dynamically complete, so whether one makes the present value of the stream $D_{t,s}$ the property of new entrepreneurs at time s or the joint property of entrepreneurs and low-skilled workers does not impact the present value of resources accruing to the cohort arriving at time s, and hence the consumption process obtained by the different generations of agents. Therefore, the interest rate r_t also remains unchanged.

By the same token, it is not important to assume that all dividend income depreciates in the same way. A further extension would have entrepreneurs introduce not only firms with depreciation rate δ_t^d, but also firms with depreciation rate δ^l. In this extension, the observed price-dividend ratio in the market would be a weighted average of q_t^d and q_t^l.

E Levered and Unlevered Equity

In our model firms are un-levered; in reality, they are levered. In order to compare the quantitative predictions of the model to the data, we need to account for this leverage, since it affects the cash flows accruing to shareholders and, by implication, the stochastic properties of returns and dividends.

To relate the excess returns of an unlevered firm to those of a levered firm, we introduce some notation. Specifically, we suppress the vintage s from the notation (since the returns, price-to-dividend ratios, etc., of all vintages are the same) and let V_t denote the total value of a given firm, S_t the value of one share of (levered) equity of this firm, n_t the number of shares outstanding, and

[19] This could be further micro-founded by assuming segmented labor markets, whereby low skilled workers of vintage s can only work in firms of vintage s.

L_t the value of debt. Since we abstract from taxes in this paper, we have

$$n_t S_t = V_t - L_t. \tag{89}$$

We will assume that the number of shares n_t is not constant, but time-varying. Specifically, the firm issues or buys back shares in so as to maintain a constant leverage ratio:

$$n_t S_t = \phi V_t, \tag{90}$$

for $0 < \phi < 1$. This simplifying assumption is standard in the macro-finance literature (see, for instance, Abel (1999) and Barro (2006)). Equation (89) can be construed as the definition of the equity value viewed as a portfolio consisting of the firm's assets and (short) debt. The return on this portfolio can be computed using either side of the equation, i.e.,

$$n_t dS_t + D_t dt = dV_t + D_t dt - L_t r_t dt. \tag{91}$$

The left-hand side records the capital gains on the n_t shares and the total dividend to equity holders; the right-hand side records the change in value of firm assets plus the total dividend, minus the net return to debt holders, i.e., the interest payment.

From (91), we derive the dynamics of S_t in terms of those of V_t, using (90):

$$\begin{aligned} \frac{dS_t}{S_t} &= \frac{dV_t - r_t L_t dt}{n_t S_t} \\ &= \frac{1}{\phi}\frac{dV_t}{V_t} - \frac{1-\phi}{\phi} r_t dt. \end{aligned} \tag{92}$$

The number of shares n_t must be adjusted to maintain the constant leverage ratio stated by (90), which means, by Ito's lemma,

$$\frac{dn_t}{n_t} = \left(\frac{V_t}{S_t}\right)^{-1} d\left(\frac{V_t}{S_t}\right) \tag{93}$$

$$= \frac{dV_t}{V_t} - \frac{dS_t}{S_t} + \sigma_{S,t}^2 - \sigma_{V,t}\sigma_{S,t}, \tag{94}$$

where $\sigma_{V,t}$ denotes the volatility of $\log(V_t)$ and $\sigma_{S,t}$ that of $\log(S_t)$. These two volatilities are related

through equation (92), which implies

$$\sigma_{S,t} = \frac{\sigma_{V,t}}{\phi}. \tag{95}$$

Plugging equations (92) and (95) in (94), we obtain the dynamics of n_t:

$$\frac{dn_t}{n_t} = -\frac{1-\phi}{\phi}\left(\frac{dV_t}{V_t} - r_t dt\right) + \frac{1-\phi}{\phi^2}\sigma_{V,t}^2 dt. \tag{96}$$

We note that the number of shares n_t in equation (96) decreases (increases) when the total value of the firm experiences a positive shock that makes the total firm value grow faster (slower) than the interest rate. This is intuitive since maintaining a constant leverage ratio requires share repurchases (issuance).

We also note that (91) gives the return-per-share as

$$\frac{dS_t + \frac{D_t}{n_t}dt}{S_t} - r_t dt = \frac{dV_t + D_t dt - (1-\phi)r_t dt}{\phi V_t} - r_t dt = \frac{1}{\phi}\left(\frac{dV_t + D_t dt}{V_t} - r_t dt\right), \tag{97}$$

which is the familiar Modigliani-Miller formula.

One useful observation is that dynamics of the dividends per share reflect both the fluctuations in total dividends and those in the number of shares. The dividend process entering the definition of the price-per-share is not the total dividend, but rather the dividend-per-share $\frac{D_t}{n_t}$. According to Ito's lemma,

$$\begin{aligned}
\left(\frac{D_t}{n_t}\right)^{-1} d\left(\frac{D_t}{n_t}\right) &= \frac{dD_t}{D_t} + \frac{dn_t^{-1}}{n_t^{-1}} \\
&= -(\delta_t^d + g)dt + \frac{1-\phi}{\phi}\left(\frac{dV_t}{V_t} - r_t dt\right) - \frac{1-\phi}{\phi^2}\sigma_{V,t}^2 dt + \frac{(1-\phi)^2}{\phi^2}\sigma_{V,t}^2 dt \\
&= \frac{1-\phi}{\phi}\left(\frac{dV_t}{V_t} - \left(r_t + \sigma_{V,t}^2\right)dt\right) - (\delta_t^d + g)dt
\end{aligned} \tag{98}$$

To summarize the results of this section, we have that i) the excess return of levered equity is $\frac{1}{\phi}$ times the return of unlevered equity, ii) the dynamics of (per-share) dividends are given by $d\left(\frac{D_t}{n_t}\right)$ — equation (98) — for levered equity and dD_t for unlevered equity and iii) the price-to-dividend ratio is $q_t = q_t^d = \frac{V_t}{D_t}$ for unlevered equity and $\frac{S_t}{D_t/n_t} = \frac{\phi V_t}{D_t} = \phi q_t$ for levered equity.

Table 1 in the main text provides the model-implied moments for levered equity, which are more

	No leverage	Leverage
Average aggregate consumption growth rate	2.3 %	2.3 %
Volatility of aggregate consumption growth rate	0 %	0 %
Individual agents' annual volatility of consumption growth	0.60%	0.60%
Sharpe ratio	0.24	0.24
Stock market volatility	9.67%	16.44%
Equity premium	2.72%	4.63%
Interest rate	0.80%	0.80%
Standard deviation of the interest rate	0.60%	0.60%
Average (log) Price-Dividend ratio	3.44	2.91
Standard deviation of the log Price-Dividend ratio	0.28	0.28
Autocorrelation of the log Price-Dividend ratio	0.92	0.92
Dividend growth volatility	3.0%	8.45%

Table 3: Model implied moments for levered and unlevered equity.

readily comparable to the data. Table 3 provides a comparison between model-implied moments for levered and un-levered equity.