

# Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing

Nicolae Gârleanu\*  
Stavros Panageas†

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## Abstract

We study the implications of preference heterogeneity for asset pricing. We use recursive preferences in order to separate heterogeneity in risk aversion from heterogeneity in the intertemporal elasticity of substitution, and an overlapping-generations framework to obtain a non-degenerate stationary equilibrium. We solve the model explicitly up to the solutions of ordinary differential equations, and highlight the effects of overlapping generations and each dimension of preference heterogeneity on the market price of risk, interest rates, and the volatility of stock returns. We find that separating IES and risk aversion heterogeneity can have a substantive impact on the model's (qualitative and quantitative) ability to address some key asset pricing issues.

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\*UC Berkeley, Haas School of Business, Centre of Economic Policy Research (CEPR), and National Bureau of Economic Research (NBER), email: garleanu@haas.berkeley.edu.

†University of Chicago, Booth School of Business and National Bureau of Economic Research (NBER), email: stavros.panageas@chicagobooth.edu. We would like to thank Andy Abel, Hengjie Ai, Nicole Branger, John Cochrane, Mike Gallmeyer, John Heaton, Leonid Kogan, Debbie Lucas, Monika Piazzesi, Kathy Yuan, four anonymous referees, and participants at seminars and presentations at Berkeley Haas, the 2008 CARESS-Cowles Conference, Chicago Booth, Drexel, the 2007 Duke UNC Asset Pricing Conference, the 2011 Finance UC Conference, HEC Paris, the 2012 ITAM Finance Conference, the 2007 NBER Summer Institute (AP), Princeton University, the 2010 Royal Economic Society, the 2007 SAET conference, the 2008 SED conference, Stanford GSB, Studienzentrums Gerzensee, the 2008 UBC Winter Finance Conference, University of Melbourne, University of Tokyo, and the 2008 UWFC for useful comments.

We study the asset pricing implications of a continuous-time, overlapping generations (OLG) model with consumers differing in terms of their intertemporal elasticity of substitution, their risk aversion or, both. We make two contributions to the literature. First, we disentangle the effects, and explain the interaction, of two conceptually distinct dimensions of preference heterogeneity, namely risk-aversion and intertemporal-elasticity-of-substitution (IES) heterogeneity. Second, the OLG feature of our framework guarantees a (non-trivial) stationary equilibrium, with all types of investors accounting for non-zero fractions of wealth and consumption in the long run. This outcome is in contrast to most of the literature on preference heterogeneity with infinitely-lived consumers, where only one group “survives” in the long run. The stationarity of our framework facilitates and simplifies comparisons between the model and empirical results.

We provide a new way of characterizing an equilibrium of OLG economies with heterogeneous recursive preferences, by reducing the solution to a system of ordinary differential equations. Besides providing a simple way to solve the model numerically, these differential equations facilitate analytical results.

We can summarize our main results as follows. First, the optimal consumption-allocation rule introduces persistent components in individual agents’ consumption growth, although *aggregate* consumption growth is i.i.d.. As a result, our consumers, who have recursive preferences, may require a higher compensation for risk (compared to the expected-utility case). We provide an analytical characterization of such cases. We also show that if one tried to infer the risk aversion of a fictitious, expected-utility-maximizing “representative” agent in our economy, the resulting measure would not correspond to a weighted average of the risk aversion coefficients of consumers in the economy; it could even exceed the risk-aversion coefficient of any investor. Second, disentangling risk-aversion heterogeneity from IES heterogeneity is important for separating variations in the interest rate from variations in the equity premium. This feature can help improve substantially the quantitative performance of the model (compared to models featuring heterogeneous agents with the same risk aversions, but expected utilities). Finally, preference heterogeneity can have a non-trivial impact on asset prices, while impacting relatively little the time variation of the consumption and wealth distributions.

Our paper relates primarily to the analytical asset-pricing literature on preference heterogeneity.<sup>1</sup> As already mentioned, this literature does not separate IES heterogeneity from risk-aversion heterogeneity. Furthermore, with the notable exception<sup>2</sup> of Chan and Kogan (2002), these models imply generically non-stationary dynamics.<sup>3</sup>

A theoretical literature has considered preference aggregation for infinitely-lived agents with heterogeneous, recursive preferences.<sup>4</sup> From a methodological perspective, we develop

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<sup>1</sup>This literature is too large to summarize here. The seminal contribution is Dumas (1989).

<sup>2</sup>In Chan and Kogan (2002) stationarity results from coupling preference heterogeneity with external habit formation. Besides making it difficult to conceptually isolate the effects of preference heterogeneity, a practical shortcoming of external habit formation is that it implies a volatile interest rate.

<sup>3</sup>We use instead an OLG approach to obtain stationarity. Spear and Srivastava (1986) is an early contribution studying stationarity in a homogenous-preference, multiple-good OLG economy.

<sup>4</sup>See, e.g., Lucas and Stokey (1984), Epstein (1987), and Dumas et al. (2000). Backus et al. (2008) gives

a novel solution approach, suitable also to OLG frameworks. It does not rely on solving a central planning problem, which works only in models without births. Furthermore, unlike in the literature, our focus is not on preference-aggregation results, but rather on developing the asset pricing implications of (recursive) preference heterogeneity.

Another relevant literature employs numerical methods to solve life-cycle-of-earnings models in general equilibrium. The paper most closely related to ours is Gomes and Michaelides (2008).<sup>5</sup> This paper obtains joint implications for asset returns and stock-market participation decisions in a rich setup that includes costly participation and heterogeneity in both preferences and income. However, for numerical tractability, the paper assumes that the volatility of output, the volatility of stock returns, and the equity premium are driven by exogenous, random capital-depreciation shocks, which produce large fluctuations in the quantity rather than the price of capital. By considering a Lucas-style endowment economy, we abstract from modeling investment, at the benefit of ensuring that stock-market fluctuations are due to endogenous variations in the price (rather than quantity) of capital. Accordingly, we can better analyze the implications of time-varying discount rates for the volatility and the equity premium. Another difference with the calibration-oriented literature is the tractability of our framework, which allows us to obtain analytical results.

We also relate to the literature that analyzes the role of OLG in asset pricing.<sup>6</sup> Many of these models combine the OLG structure with other frictions or shocks to drive incomplete risk sharing across generations, so that consumption risk is disproportionately high for cohorts participating predominantly in asset markets. Even though we think these channels important for asset pricing, we do not include them in order to isolate the intuitions pertaining to preference heterogeneity.

## 1 Model

### 1.1 Agents, firms, and endowments

Our specification of demographics follows Blanchard (1985). Time is continuous. Each agent faces a constant hazard rate of death  $\pi > 0$  throughout her life, so that a fraction  $\pi$  of the population perishes. A new cohort of mass  $\pi$  is born per unit of time, so that the total population size is  $\int_{-\infty}^t \pi e^{-\pi(t-s)} ds = 1$ .

To allow for the separation of the effects of the IES and the risk aversion, we assume that agents have the type of Kreps-Porteus-Epstein-Zin-Weil recursive preferences adapted by Duffie and Epstein (1992) to continuous-time settings. Specifically, an agent of type  $i$

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a closed-form solution for a special limiting case. We note that, unlike in Branger et al. (2012), survival in our model is generic. In particular, we can analyze parameter constellations (e.g., the heterogeneous CRRA case) incompatible with stationarity when agents are infinitely lived.

<sup>5</sup>Other related work includes Storesletten et al. (2007) and Guvenen (2009).

<sup>6</sup>Indicative examples of such papers are Constantinides et al. (2002), Piazzesi and Schneider (2009), Gârleanu et al. (2012). Farmer (2002) analyzes the case of homogeneous, logarithmic preferences.

maximizes her utility  $V^i$  given by

$$V_s^i = \mathbb{E}_s \left[ \int_s^\infty f^i(c_u^i, V_u^i) du \right], \quad (1)$$

$$f^i(c, V) \equiv (\alpha^i)^{-1} \left( (1 - \gamma^i) V \right)^{1 - \frac{\alpha^i}{1 - \gamma^i}} \left( c^{\alpha^i} - (\rho + \pi) \left( (1 - \gamma^i) V \right)^{\frac{\alpha^i}{1 - \gamma^i}} \right). \quad (2)$$

The function  $f^i(c, V)$  aggregates the utility arising from current consumption  $c$  and the value function  $V$ . The parameter  $\gamma^i > 0$  controls the risk aversion of agent  $i$ , while  $(1 - \alpha^i)^{-1}$  gives the agent's IES. We assume that  $\alpha^i < 1$ , and note that when  $\alpha^i = 1 - \gamma^i$  these utilities reduce to standard constant-relative-risk aversion (CRRA) utilities. The parameter  $\rho$  is the agent's subjective discount factor. The (online) appendix,<sup>7</sup> section D, gives a short derivation of the objective function (1) as the continuous-time limit of a discrete-time, recursive-preference specification with random times of death.

To study the effects of heterogeneity in the most parsimonious way, we assume that there are two types of agents, labeled  $A$  and  $B$ . At every point in time a proportion  $v^A \in (0, 1)$  of newly born agents are of type  $A$  and  $v^B = 1 - v^A$  are of type  $B$ . For the rest of the paper we maintain the convention  $\gamma^A \leq \gamma^B$ .

Aggregate output  $Y_t$  evolves as  $dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dB_t$ , where  $\mu_Y$  and  $\sigma_Y$  are parameters and  $B_t$  is a standard Brownian motion. At time  $t$ , an agent born at time  $s$  is endowed with earnings  $y_{t,s}$ , where  $y_{t,s} = \omega Y_t G(t - s)$  for  $\omega \in (0, 1)$  and  $G \geq 0$  is a function of age that controls the life-cycle earnings profile. In Sections 3 and 4 we specialize  $G$  to

$$G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}. \quad (3)$$

This parametric form is flexible enough to reproduce the hump-shaped pattern of earnings observed in the data, as we illustrate in the appendix (Section B). Aggregate earnings equal

$$\int_{-\infty}^t \pi e^{-\pi(t-s)} y_{t,s} ds = \omega Y_t \int_{-\infty}^t \pi e^{-\pi(t-s)} G(t-s) ds = \omega Y_t \int_0^\infty \pi e^{-\pi u} G(u) du. \quad (4)$$

We assume throughout that  $\int_0^\infty \pi e^{-\pi u} G(u) du < \infty$ , and normalize the value of this integral to 1 by scaling  $G$ . The aggregate earnings are therefore given by  $\omega Y_t$ , the remaining fraction  $1 - \omega$  of output being paid out as dividends  $D_t \equiv (1 - \omega) Y_t$  by the representative firm.

## 1.2 Markets and budget constraints

Agents can allocate their portfolios between shares of the representative firm and instantaneously maturing riskless bonds, which pay an interest rate  $r_t$ . The supply of shares of the firm is normalized to one, while bonds are in zero net supply. The price  $S_t$  of each share evolves according to  $dS_t = (\mu_t S_t - D_t) dt + \sigma_t S_t dB_t$ , where the coefficients  $\mu_t$  and  $\sigma_t$  are determined in equilibrium, as is the interest rate  $r_t$ .

<sup>7</sup>Available at <http://faculty.chicagobooth.edu/stavros.panageas/research/Appendices.pdf>.

Finally, agents can access a market for annuities through competitive insurance companies as in Blanchard (1985), allowing them to receive an income stream of  $\pi W_{t,s}$  per unit of time. In exchange, the insurance company collects the agent's financial wealth when she dies. Entering such a contract is optimal for all agents, given the absence of bequest motives.

Letting  $\theta_{t,s}$  denote the dollar amount invested in shares of the representative firm,<sup>8</sup> an agent's financial wealth  $W_{t,s}$  consequently evolves according to

$$dW_{t,s} = (r_t W_{t,s} + \theta_{t,s} (\mu_t - r_t) + y_{t,s} + \pi W_{t,s} - c_{t,s}) dt + \theta_{t,s} \sigma_t dB_t, \quad W_{s,s} = 0. \quad (5)$$

### 1.3 Equilibrium

The definition of equilibrium is standard. An equilibrium is given by a set of adapted processes  $\{c_{t,s}, \theta_{t,s}, r_t, \mu_t, \sigma_t\}$  such that (i) the processes  $c_{t,s}$  and  $\theta_{t,s}$  maximize an agent's objective (1) subject to the dynamic budget constraint (5), and (ii) markets for goods clear, i.e.,  $\int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s} ds = Y_t$ , and markets for stocks and bonds clear as well:  $\int_{-\infty}^t \pi e^{-\pi(t-s)} \theta_{t,s} ds = S_t$  and  $\int_{-\infty}^t \pi e^{-\pi(t-s)} (W_{t,s} - \theta_{t,s}) ds = 0$ .

## 2 Homogeneous preferences

Our model setup has two main features: OLG and preference heterogeneity. In preparation for the main results of the paper, this section clarifies the asset-pricing implications of OLG (absent preference heterogeneity).

**Proposition 1** *Suppose that all agents have the same preferences ( $\alpha^A = \alpha^B = \alpha$ ,  $\gamma^A = \gamma^B = \gamma$ ), and consider the following non-linear equation for  $r$ :*

$$r = \rho + (1 - \alpha) (\mu_Y + \pi (1 - \beta(r))) - \gamma (2 - \alpha) \frac{\sigma_Y^2}{2}, \quad (6)$$

$$\beta(r) = \omega \left( \int_0^\infty G(u) e^{-(r + \pi + \gamma \sigma_Y^2 - \mu_Y)u} du \right) \left( \pi + \frac{\rho}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \left( r + \frac{\gamma}{2} \sigma_Y^2 \right) \right). \quad (7)$$

*Suppose that  $\bar{r}$  is a root of (6) and  $\bar{r} > \mu_Y - \gamma \sigma_Y^2$ .<sup>9</sup> Then there exists an equilibrium in which the interest rate, the expected return, and the volatility of the stock market are all constant and given, respectively, by  $r_t = \bar{r}$ ,  $\mu_t = \bar{r} + \gamma \sigma_Y^2$ , and  $\sigma_t = \sigma_Y$ .*

Clearly, the OLG feature does not help address issues such as the equity premium, the excess volatility puzzle, the predictability of returns, etc. Its only interesting asset-pricing implication is an interest rate that differs from its value in the respective infinitely-lived representative-agent economy, namely  $\rho + (1 - \alpha) \mu_Y - \gamma (2 - \alpha) \sigma_Y^2 / 2$ , due to the additional term  $(1 - \alpha) \pi (1 - \beta)$ .

<sup>8</sup>We assume standard square integrability and transversality conditions. See, e.g., Karatzas and Shreve (1998) for details.

<sup>9</sup>Lemma 2 in the appendix (section A) contains parameter conditions sufficient for such a root to exist.

The source of the difference is that in an OLG economy only the Euler equation (and hence the per-capita consumption growth) of *existing* agents matters. The per-capita consumption growth of existing agents is given by  $\mu_Y + \pi(1 - c_{t,t}/C_t)$ , which is in general different from the growth rate  $\mu_Y$  of *aggregate* consumption. Death raises the per-capita consumption growth rate of surviving agents above the aggregate consumption growth rate by the death rate  $\pi$ , while births lower it by the product of the birth rate  $\pi$  and the fraction of aggregate consumption accruing to newly born agents,  $c_{t,t}/C_t$ .

As we show in the appendix, when all agents have homogeneous preferences,  $c_{t,t}/C_t$  is constant and equal to the constant  $\beta$  given in Proposition 1. In the appendix (Lemma 2), we also extend some results of Blanchard (1985) and show that under certain conditions — most important, earnings are generally lower late in life — newly born agents consume more than existing ones:  $\beta > 1$ . Consequently, the interest rate in our economy (equation (6)) is lower than in the respective economy featuring an infinitely-lived agent, thus helping address the low risk-free rate puzzle.<sup>10</sup>

### 3 Heterogeneous preferences

We can now turn to the main results of the paper, by allowing agents to be heterogeneous ( $\gamma^A \neq \gamma^B$ ,  $\alpha^A \neq \alpha^B$ ). We describe an equilibrium in which the interest rate  $r_t$  and the Sharpe ratio  $\kappa_t \equiv (\mu_t - r_t)/\sigma_t$  are functions of the consumption share of type-A agents,

$$X_t \equiv v^A \pi \int_{-\infty}^t e^{-\pi(t-s)} c_{t,s}^A ds \times Y_t^{-1}. \quad (8)$$

In such an equilibrium,  $X_t$  behaves like a Markovian diffusion process. Moreover, there are two important valuation ratios that become exclusively functions of  $X_t$ . The first ratio is the consumption-to-wealth ratio, denoted by  $g^i(X_t)$ , and the second is the ratio of an agent's present value of earnings (thus total wealth) at birth to aggregate consumption, denoted by  $\phi(X_t)$ . The functions  $g^i$  and  $\phi$  are determined in equilibrium. The following result holds.

**Proposition 2** *Let  $G$  be given by (3) and  $\beta_t^i \equiv c_{t,t}^i/Y_t$  denote the consumption of a newly-born agent of type  $i$  as a fraction of output. Let  $X_t^A \equiv X_t$ ,  $X_t^B \equiv 1 - X_t$ ,  $\Gamma(X_t) \equiv (\sum_i X_t^i/\gamma^i)^{-1}$ ,  $\Theta(X_t) \equiv \sum_i X_t^i/(1 - \alpha^i)$ ,  $\omega^i(X_t) \equiv X_t^i \Gamma(X_t)/\gamma^i$ , and  $\Delta(X_t) \equiv \sum_i \omega^i(X_t)(\gamma^i + 1)/\gamma^i$ . Finally, assume functions  $g^i(X_t)$  and  $\phi(X_t)$  that solve the system of ordinary differential equations (A.19) and (A.22) in the appendix. Then there exists an equilibrium in which  $X_t$  is a Markov diffusion with dynamics  $dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dB_t$ . We have*

$$\sigma_X(X_t) = \frac{X_t (\Gamma(X_t) - \gamma^A)}{\frac{\Gamma(X_t)}{\gamma^B} X_t (1 - X_t) \left[ \frac{1 - \gamma^A - \alpha^A}{\alpha^A} \frac{g^{A'}}{g^A} - \frac{1 - \gamma^B - \alpha^B}{\alpha^B} \frac{g^{B'}}{g^B} \right] + \gamma^A} \sigma_Y, \quad (9)$$

<sup>10</sup>In the appendix (Remark 2) we quantify this difference and show that the interest rate in the OLG economy is low even for low IES levels, which generate particularly high interest rates in the infinitely-lived-agent economy.

$$\mu_X(X_t) = X_t \left[ \frac{r(X_t) - \rho}{1 - \alpha^A} + n^A(X_t) - \pi - \mu_Y \right] + v^A \pi \beta^A(X_t) - \sigma_Y \sigma_X(X_t), \quad (10)$$

$$\kappa(X_t) = \Gamma(X_t) \sigma_Y + \sum_i \omega^i(X_t) \left( \frac{1 - \gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \sigma_X(X_t), \quad (11)$$

$$r(X_t) = \rho + \frac{1}{\Theta(X_t)} \left\{ \mu_Y - \pi \left( \sum_i v^i \beta^i(X_t) - 1 \right) \right\} - \frac{1}{\Theta(X_t)} \sum_i X_t^i n^i(X_t), \quad (12)$$

where  $\beta^i(X_t) = g^i(X_t)\phi(X_t)$  and the functions  $n^i(X_t)$  are given explicitly in the appendix.

Proposition 2 is the main proposition of the paper. In preparation for the discussion of the asset-pricing implications of the model, it is useful to note some properties of  $X_t$ .

Unlike models featuring infinitely lived agents, our setup implies (generically) that no group of agents becomes extinct in the long run, in the sense that the process for  $X_t$  does not get absorbed at the points 0 or 1.<sup>11</sup> Intuitively, although the mean growth rates of wealth differ across agents, eventually all agents perish. All newly-born agents enter the economy with human capital that is equal across preference groups and proportional to the level of output. As a result, each group of agents receives a minimum inflow of new members whose consumption is a non-zero fraction of aggregate output, ensuring that no group of agents dominates the economy in the long run.

An additional interesting implication of Proposition 2 is that risk-aversion heterogeneity is necessary for  $X_t$  to be stochastic. Specifically, we obtain the following corollary.

**Corollary 1** *Consider the setup of Proposition 2 and impose the restriction  $\gamma^A = \gamma^B = \gamma$ . Then  $\sigma_X = 0$  at all times, the market price of risk is given by the constant  $\kappa = \gamma\sigma$  for all  $X_t$ , and there exists a steady state featuring a constant interest rate and a constant consumption share of type-A agents given by some value  $\bar{X} \in (0, 1)$ .*

Corollary 1 shows that risk-aversion heterogeneity is essential for the model to have any interesting asset-pricing implications; for the rest of the paper we assume  $\gamma^A < \gamma^B$ .

### 3.1 Market price of risk ( $\kappa_t$ )

In the case of heterogeneous CRRA preferences ( $1 - \alpha^i = \gamma^i$ ) equations (11) and (9) specialize to  $\kappa(X_t) = \Gamma(X_t)\sigma_Y$  and  $\sigma_X(X_t) = X_t \left( \frac{\Gamma(X_t)}{\gamma^A} - 1 \right) > 0$ . Hence, the market price of risk is determined as if the economy is populated by a representative agent with risk aversion  $\Gamma(X_t)$ . The risk aversion of the representative agent is time varying, countercyclical (since  $\Gamma'(X_t) < 0$  and  $\sigma_X > 0$ ), and bounded by  $\gamma_A$  and  $\gamma_B$ . The intuition is that, in response to

<sup>11</sup>To establish this result, we start by noting that equation (9) implies  $\sigma_X(0) = 0$ , while equation (10) gives  $\mu_X(0) = v^A \pi \beta^A(0)$ . Since the value of life-time earnings of any newly born agent is strictly positive, so is any newly-born agent's consumption. Thus,  $\beta^A(X_t) = c_{t,t}/Y_t > 0$  and, as a consequence,  $\mu_X(0) > 0$ . Similarly,  $\sigma_X(1) = 0$  and  $\mu_X(1) = -v^B \pi \beta^B(1) < 0$ . We note that by using results from diffusion theory, equations (9)–(12) can be used to derive an expression for the stationary distribution of  $X_t$ .



a positive (negative) aggregate shock, the wealth distribution tilts towards the less (more) risk averse consumers, who require a relatively lower (higher) compensation for risk.

A novel implication of equation (11) is that when agents have recursive preferences ( $1 - \alpha^i \neq \gamma^i$  for some  $i \in \{A, B\}$ ), the ratio  $\kappa(X_t)/\sigma_Y$  is no longer equal to  $\Gamma(X_t)$ . Instead, it contains the additional term  $\sum_i \omega^i(X_t) \left( \frac{1-\gamma^i-\alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y}$ . This term depends on (i) the weights  $\omega^i(X_t)$  of each agent, (ii) each agent's preference for late or early resolution of uncertainty ( $1 - \gamma^i - \alpha^i$ ), (iii) the preference parameters  $\alpha^i$ , which control agents' IES, and (iv) the term  $\frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y}$ , which — as we show in the appendix — captures the volatility of each agent's consumption-to-total-wealth ratio normalized by the volatility of consumption.

It follows that  $\kappa(X_t)/\sigma_Y$  no longer equals a simple weighted average of individual risk aversions. Indeed, one can construct numerical examples (we provide one such example in the appendix) in which  $\kappa(X_t)/\sigma_Y$  exceeds the risk aversion of any agent in the economy. In such a situation, a researcher using a standard, expected-utility-maximizing framework to infer the risk aversion of the “representative agent” (for such an exercise see, e.g., Ait-Sahalia and Lo (2000)) would obtain an estimate exceeding any agent's risk aversion.

Since many standard asset-pricing models tend to produce a low market price of risk for plausible assumptions on agents' risk aversions, we are particularly interested in determining conditions that ensure that the term  $\sum_i \omega^i(X_t) \left( \frac{1-\gamma^i-\alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y}$  in equation (11) is positive, so that the market price of risk  $\kappa(X_t)$  exceeds the Sharpe ratio  $\Gamma(X_t) \sigma_Y$  that one would obtain with heterogeneous, but expected-utility maximizing, agents. The following proposition addresses this question.

**Proposition 3** *Consider the setup of Proposition 2 with the parameters restricted to lie in a (arbitrary) compact set, and let  $\bar{X}$  denote the stationary mean of  $X_t$ . Then — subject to technical parameter restrictions given in Appendix A — for sufficiently small  $|\alpha^B - \alpha^A|$  and  $|\gamma^B - \gamma^A|$  the Sharpe ratio satisfies  $\kappa(\bar{X}) \geq \Gamma(\bar{X}) \sigma_Y$  if either*

- (i)  $\gamma^i + \alpha^i - 1 > 0$  for  $i \in \{A, B\}$  and  $\alpha^A < \alpha^B$ , or
- (ii)  $\gamma^i + \alpha^i - 1 < 0$  for  $i \in \{A, B\}$  and  $\alpha^A > \alpha^B$ .

The intuition behind Proposition 3 is as follows. Even though aggregate consumption growth is i.i.d., *individual* consumption growth is not. The optimal risk-sharing rule introduces predictable components in the consumption growth of the two preference groups. When coupled with appropriate assumptions on consumer preferences for early or late resolution of uncertainty, the market price of risk is increased. For instance, in case (i) individual consumers' risk aversion is higher than the inverse of the IES (preferences for early resolution of uncertainty), while their individual consumption growth is positively autocorrelated.<sup>12</sup> As in the well-known model of Bansal and Yaron (2004), this implies a higher price of risk. (The argument for case (ii) is symmetric.) An important difference between Bansal and Yaron (2004) and this paper is that predictable consumption components are not assumed exogenously at the aggregate level, but rather arise endogenously at the individual level, as part of the equilibrium consumption allocation.

<sup>12</sup>Appendix A (Remark 3) contains a detailed discussion on the source of this positive autocorrelation.



### 3.2 The interest rate ( $r_t$ )

In Section 2 we discussed why the OLG structure of our economy (absent preference heterogeneity) can lead to a lower level of the interest rate than in the respective infinite-horizon economy. With preference heterogeneity the interest rate becomes time varying. Motivated by the stylized facts listed in Campbell and Cochrane (1999), we are particularly interested whether there exist constellations of preference parameters for which the volatility of discount rates is due almost exclusively to equity-premium, rather than interest-rate, variability.

We can provide an affirmative answer to this question using an approximation (accurate when  $\sigma_Y$  is not too large) around the stationary mean of  $X_t$ .

**Proposition 4** *Fix  $D_1 > 0$  and  $D_2 > 0$ . Then there exist parameters  $\gamma^A$ ,  $\gamma^B$ , and  $v^A$ , as well as  $\alpha^A$  and  $\alpha^B$  possibly depending on  $\sigma_Y$ , so that at the steady-state mean  $\bar{X}$ ,*

$$\frac{\kappa(\bar{X})}{\sigma_Y} = D_1 + \epsilon_1(\sigma_Y^2), \quad -\frac{\kappa'(\bar{X})}{\sigma_Y} = D_2 + \epsilon_2(\sigma_Y^2), \quad \frac{r'(\bar{X})}{\sigma_Y^2} = \epsilon_3(\sigma_Y^2), \quad (13)$$

where the terms  $\epsilon_i(\sigma_Y^2)$  satisfy  $\lim_{x \rightarrow 0} \epsilon_i(x) = 0$  for  $i \in \{1, 2, 3\}$ . Accordingly, the relative magnitude of the volatility of the equity premium to the volatility of the interest rate,  $\frac{|\kappa(\bar{X})\sigma(\bar{X})'|_{\sigma_X(\bar{X})}}{|r'(\bar{X})|_{\sigma_X(\bar{X})}}$ , can be made arbitrarily large.

The main thrust of Proposition 4 is that one can determine a joint distribution of IES and risk aversion so as to ensure (around the stationary mean of  $X_t$ ) that the interest rate is substantially less volatile than the equity premium. The intuition behind this result is as follows. If investors had equal IES, then the redistribution of wealth towards the less risk-averse agents induced by a positive shock would cause both the market price of risk to fall and the interest rate to rise — because the precautionary savings would fall. By increasing appropriately the IES of the less risk-averse agents, therefore, the interest rate can be rendered insensitive to output shocks, without affecting (beyond the second order) the sensitivity of the Sharpe ratio.

## 4 Quantitative implications

In this section we illustrate the quantitative implications of the above propositions by numerically determining combinations of IES and risk aversion associated with a high equity premium, volatile returns, and a low and non-volatile interest rate. We provide details of how we specify all the parameters in the appendix (Section B). Here we give a brief summary: We choose  $\mu_Y$  and  $\sigma_Y$  to match the first two moments of aggregate, time-integrated consumption growth,  $\pi$  to match the birth rate, and  $\omega$  to match the fraction of capital income in national income.  $B_1$ ,  $B_2$ ,  $\delta_1$ , and  $\delta_2$  are estimated by regressing the function  $G$  in (3) on the empirical life-cycle profile of earnings. We set  $\rho = 0.001$ . With this choice, the investors' "effective" discount rate is  $\rho + \pi \approx 0.02$ . We treat the remaining parameters, namely the risk aversions

Parameter	Data	Baseline	CRRA
Risk aversion of type- <i>A</i> agents		1.5	1.5
Risk aversion of type- <i>B</i> agents		10	10
IES of type- <i>A</i> agents		0.7	$(\gamma^A)^{-1}$
IES of type- <i>B</i> agents		0.05	$(\gamma^B)^{-1}$
Population share of type- <i>A</i> agents		0.01	0.01
Equity premium	5.2 %	5.1 %	2.1 %
Volatility of returns	18.2%	17.0 %	8.5 %
Average (real) interest rate	2.8%	1.4%	2.1 %
Volatility of the real interest rate	0.92%	0.93%	0.13 %

Table 1: Unconditional moments for the data, the baseline parametrization and a CRRA parametrization (restricting the IES to be the inverse of the relative risk aversion). The data for the average equity premium, the volatility of returns, and the level of the interest rate are from the long historical sample (1871-2011) available from the website of R. Shiller (<http://www.econ.yale.edu/~shiller/data/chapt26.xls>). The volatility of the “real rate” is inferred from the yields of 5-year constant maturity TIPS.

and IES, as well as the share of type-*A* agents in the population ( $v^A$ ) as free parameters. In the spirit of Proposition 4, we choose these parameters to approximately match asset-pricing moments and, in particular, the level of the equity premium, the volatility of returns, and the level and volatility of the interest rate. To keep the exercise empirically relevant, we place an upper bound on the highest risk aversion, so that  $\gamma^B = 10$ .<sup>13</sup> Clearly, this constraint limits the ability of the model to match all moments exactly.

In reporting the results we follow the approach of Barro (2006) to relate the results of our model (which produces implications for an all-equity financed firm) to the data (where equity is levered). Specifically, we use the well known Modigliani-Miller formula relating the returns of levered equity to those of unlevered equity, along with the historically observed debt-to-equity ratio, to report model-implied levered returns.

Table 1 provides a comparison between the model and the data in terms of unconditional moments. We note that the flexibility to choose agents’ IES independent of their risk aversion is important for the model’s quantitative implications. Without this flexibility the model doesn’t perform as well, as illustrated in the third column of the table. Interestingly, the main reason why the baseline model performs better is *not* its higher Sharpe ratio, which only increases from 0.25 to 0.3 when we compare the heterogeneous CRRA case to the baseline (heterogeneous recursive-preference) case. Rather, the main reason for the better performance of the baseline model is the higher volatility of returns compared to the heterogeneous-CRRA case. The intuition is the following. In the baseline calibration the variation in the interest rate and the equity premium tend to reinforce each other in the sense that both variables decline and rise together. By contrast, in the CRRA case the variation in the interest rate tends to largely offset the variation in the equity premium over a significant part of the state

<sup>13</sup>This upper bound is common in the literature. See, e.g. Mehra and Prescott (1985).

Horizon (Years)	Data (Long Sample)		Model	
	Coefficient	$R^2$	Coefficient	$R^2$
1	-0.13	0.04	-0.09 [-0.29,0.01]	0.03 [0.00,0.10]
3	-0.35	0.09	-0.25 [-0.68,0.00]	0.08 [0.00,0.23]
5	-0.60	0.18	-0.41 [-1.03,0.01]	0.12 [0.00,0.32]
7	-0.75	0.23	-0.54 [-1.27,0.00]	0.16 [0.00,0.41]

Table 2: Long-horizon regressions of excess returns on the log P/D ratio. The simulated data are based on 1000 independent simulations of 106-year long samples, where the initial condition for  $X_0$  for each of these simulation paths is drawn from the stationary distribution of  $X_t$ . For each of these 106-year long simulated samples, we run predictive regressions of the form  $\log R_{t \rightarrow t+h}^e = \alpha + \beta \log(P_t/D_t)$ , where  $R_{t \rightarrow t+h}^e$  denotes the time- $t$  gross excess return over the next  $h$  years. We report the median values for the coefficient  $\beta$  and the  $R^2$  of these regressions, along with the respective [0.025, 0.975] percentiles.

space, thus mitigating the overall countercyclical behavior of the total discount rate.

Besides matching unconditional moments of the data, the model can also match well known facts pertaining to the predictability of excess returns by such predictive variables as the price-to-dividend ratio. Table 2 shows that the model reproduces the empirical evidence on predictability of excess returns over long horizons. This is not surprising in light of Figure 1: Since dividend growth is i.i.d. in our model, movements in the price-dividend ratio must be predictive of changes in discount rates. As Figure 1 shows, interest rate movements in our model are of much smaller magnitude than excess-return movements. As a result the price-dividend ratio predicts excess returns quite well. In results not reported here<sup>14</sup> we also confirm that the model yields the empirically correct sign and magnitude in predictive regressions of interest rates on the price-dividend ratio.

Beyond asset pricing, our parameters have implications for the cross-sectional distributions of consumption and wealth. We use the Gini coefficient to summarize the inequality associated with these distributions. The Gini coefficient for consumption is 0.3 in the model (as compared to 0.28 in the data), while the respective Gini coefficient for wealth inequality is 0.67 in the model (0.9 in the data). Importantly, the model does not lead to large variations in the annual volatility of consumption inequality (the standard deviation of annual changes in the consumption Gini coefficient is 0.95 in the model and 0.7 in the data).

Finally, we note that the IES parameters in the baseline calibration are also in line with the findings of empirical studies utilizing micro-level data to estimate Euler equations.

<sup>14</sup>These results are in Section C of the appendix, along with a discussion of the “path-wise” behavior of the model. Specifically, we compare historical returns with the returns implied by the model when the consumption shocks are as observed in the data.

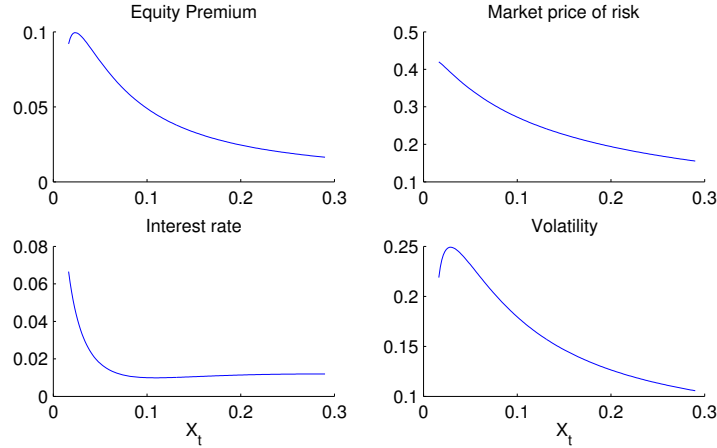


Figure 1: Equity premium, market price of risk, interest rate, and return volatility for the baseline parametrization as a function of the consumption share of type- $A$  agents ( $X_t$ ). The range of values of  $X_t$  is the interval between the 0.5 and the 99.5 percentiles of the stationary distribution of  $X_t$ .

A stylized fact of such studies is that IES estimates for poorer households (the type “B” agents in the model) tend to be close to zero, while those for richer households tend to be substantially higher — consistent with the IES specifications in our baseline quantification.<sup>15</sup>

## 5 Concluding remarks

We disentangle the effects of IES and risk-aversion heterogeneity in an OLG framework that ensures stationary consumption and wealth distributions. We provide conditions ensuring that the market price of risk is higher if agents have recursive, rather than time-separable, preferences, for the same risk-aversion coefficients. Furthermore, we find that the ability to separate IES heterogeneity from risk-aversion heterogeneity plays an important role in disentangling variation in interest rates from variation in risk premia, which is crucial for the model’s ability to (quantitatively) address a host of asset pricing puzzles.

## References

- Ait-Sahalia, Y. and A. W. Lo (2000). Nonparametric risk management and implied risk aversion. *Journal of Econometrics* 94, 9–51.
- Backus, D., B. R. Routledge, and S. E. Zin (2008). Who holds risky assets? mimeo, NYU.
- Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance* 59(4), 1481–1509.
- Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics* 121(3), 823–66.

<sup>15</sup>These statements hold also conditional on participation. See Vissing-Jorgensen (2002).

- Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy* 93(2), 223–247.
- Branger, N., I. Dumitrescu, V. Ivanova, and C. Schlag (2012). Preference heterogeneity and survival in long-run risk models. Working paper, University of Frankfurt.
- Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy* 107(2), 205–251.
- Chan, Y. L. and L. Kogan (2002). Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices. *Journal of Political Economy* 110(6), 1255–1285.
- Constantinides, G. M., J. B. Donaldson, and R. Mehra (2002). Junior can't borrow: A new perspective of the equity premium puzzle. *Quarterly Journal of Economics* 117(1), 269–96.
- Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica* 60(2), 353–394.
- Dumas, B. (1989). Two-person dynamic equilibrium in the capital market. *Review of Financial Studies* 2(2), 157–188.
- Dumas, B., R. Uppal, and T. Wang (2000). Efficient intertemporal allocations with recursive utility. *Journal of Economic Theory* 93, 240–259.
- Epstein, L. G. (1987). A simple dynamic general equilibrium model. *Journal of Economic Theory* 41(1), 68–95.
- Farmer, R. E. A. (2002). Fiscal policy, equity premia and heterogeneous agents. Working Paper, UCLA.
- Gârleanu, N., L. Kogan, and S. Panageas (2012). The demographics of innovation and asset returns. *Journal of Financial Economics* 105, 491–510.
- Gomes, F. and A. Michaelides (2008). Asset pricing with limited risk sharing and heterogeneous agents. *Review of Financial Studies* 21(1), 415–449.
- Guvenen, F. (2009). A parsimonious macroeconomic model for asset pricing. *Econometrica* 77(6), 1711–1740.
- Karatzas, I. and S. E. Shreve (1998). *Methods of mathematical finance*, Volume 39. Springer-Verlag.
- Lucas, Robert E., J. and N. L. Stokey (1984). Optimal growth with many consumers. *Journal of Economic Theory* 32(1), 139–171.
- Mehra, R. and E. C. Prescott (1985). The equity premium: A puzzle. *Journal of Monetary Economics* 15(2), 145–161.
- Piazzesi, M. and M. Schneider (2009). Inflation and the price of real assets. Federal Reserve Bank of Minneapolis, Staff Report: 423.
- Spear, S. E. and S. Srivastava (1986). Markov rational expectations equilibria in an overlapping generations model. *Journal of Economic Theory* 38(1), 35–62.
- Storesletten, K., C. I. Telmer, and A. Yaron (2007). Asset pricing with idiosyncratic risk and overlapping generations. *Review of Economic Dynamics* 10(4), 519–548.
- Vissing-Jorgensen, A. (2002). Limited asset market participation and the elasticity of intertemporal substitution. *Journal of Political Economy* 110(4), 825–853.

# Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing

## Online Appendix

### A Proofs

We start with the proof of Proposition 2. We then provide the proofs of Proposition 1 and Corollary 1 as special cases.

**Proof of Proposition 2.** To help with the derivations, we first define the constants

$$\Xi_1^i = -\frac{\alpha^i + \gamma^i - 1}{\alpha^i}, \quad \Xi_2^i = \frac{\alpha^i}{(1 - \alpha^i)(1 - \gamma^i)}, \quad \Xi_3^i = -\frac{\rho + \pi}{\alpha^i}(1 - \gamma^i), \quad \Xi_4^i = -\frac{\alpha^i + \gamma^i - 1}{(1 - \alpha^i)(1 - \gamma^i)},$$

for  $i \in A, B$ .

Since agents can trade in a stock and a bond continuously, they face dynamically complete markets over their lifetimes. We let  $\xi_t$  denote the stochastic discount factor and further define the related process  $\tilde{\xi}$ :

$$\frac{d\xi_t}{\xi_t} \equiv -r_t dt - \kappa_t dB_t \tag{A.1}$$

$$\frac{\tilde{\xi}_u}{\tilde{\xi}_t} \equiv e^{-\pi(u-t)} \frac{\xi_u}{\xi_t}. \tag{A.2}$$

The results in Duffie and Epstein (1992) and Schroder and Skiadas (1999) imply that the optimal consumption process for agents with recursive preferences of the form (2) is given by

$$\frac{c_{u,s}^i}{c_{t,s}^i} = e^{\frac{1}{1-\alpha^i} \int_t^u f_V^i(w) dw} \left( \frac{(1-\gamma)V_{u,s}^i}{(1-\gamma)V_{t,s}^i} \right)^{\Xi_4^i} \left( \frac{\tilde{\xi}_u}{\tilde{\xi}_t} \right)^{\frac{1}{\alpha^i-1}}. \tag{A.3}$$

From this point onwards, we proceed by employing a “guess and verify” approach. First, we guess that both  $r_t$  and  $\kappa_t$  are functions of  $X_t$  and that  $X_t$  is Markov. Later we verify these conjectures.

Under the conjecture that both  $r_t$  and  $\kappa_t$  are functions of  $X_t$  and that  $X_t$  is Markov, the homogeneity of the recursive preferences in equation (2) implies that there exist a pair of

appropriate functions  $g^i(X_t)$ ,  $i \in \{A, B\}$ , such that the time- $t$  value function of an agent of type  $i$  born at time  $s \leq t$  is given by

$$V_{t,s}^i = \frac{\left(\widetilde{W}_{t,s}^i\right)^{1-\gamma^i}}{1-\gamma^i} g^i(X_t)^{\frac{(1-\gamma^i)(\alpha^i-1)}{\alpha^i}}. \quad (\text{A.4})$$

$\widetilde{W}_{t,s}^i$  denotes the total wealth of the agent given by the sum of her financial wealth and the net present value of her earnings:  $\widetilde{W}_{t,s}^i \equiv W_{t,s}^i + E_t \int_t^\infty \frac{\xi_u}{\xi_t} y_{u,s} du$ . Using (A.4) along with the first order condition for optimal consumption  $V_W = f_c$  gives  $c_{t,s}^i = \widetilde{W}_{t,s}^i g^i(X_t)$ . Using this last identity inside (A.4) and re-arranging gives

$$V_{t,s}^i = \frac{c_{t,s}^{1-\gamma^i}}{1-\gamma^i} g(X_t)^{-\frac{1-\gamma^i}{\alpha^i}}. \quad (\text{A.5})$$

Combining Equations (A.5) and (A.3) gives

$$\left(\frac{(1-\gamma^i)V_{u,s}^i}{(1-\gamma^i)V_{t,s}^i}\right)^{\frac{1}{1-\gamma^i}} \left(\frac{g(X_u)}{g(X_t)}\right)^{\frac{1}{\alpha^i}} = e^{\frac{1}{1-\alpha^i} \int_t^u f_V^i(w) dw} \left(\frac{(1-\gamma)V_{u,s}^i}{(1-\gamma)V_{t,s}^i}\right)^{\Xi_4^i} \left(\frac{\xi_u}{\xi_t}\right)^{\frac{1}{\alpha^i-1}}. \quad (\text{A.6})$$

Using the definition of  $f$  and Equation (A.5), we obtain

$$f_V^i(t) = \Xi_1^i g^i(X_t) + \Xi_3^i. \quad (\text{A.7})$$

Equation (A.7) implies that  $f_V^i(t)$  is exclusively a function of  $X_t$ . Hence Equation (A.6) implies that  $\frac{(1-\gamma^i)V_{u,s}^i}{(1-\gamma^i)V_{t,s}^i}$  is independent of  $s$ . In turn, Equation (A.3) implies that  $\frac{c_{u,s}^i}{c_{t,s}^i}$  is independent of  $s$ . Motivated by these observations, henceforth we use the simpler notation  $\frac{(1-\gamma^i)V_u^i}{(1-\gamma^i)V_t^i}$  and  $\frac{c_u^i}{c_t^i}$  instead of  $\frac{(1-\gamma^i)V_{u,s}^i}{(1-\gamma^i)V_{t,s}^i}$  and  $\frac{c_{u,s}^i}{c_{t,s}^i}$ , respectively.

Solving for  $\frac{(1-\gamma^i)V_u^i}{(1-\gamma^i)V_t^i}$  from Equation (A.6) and applying Ito's Lemma to the resulting equation gives

$$d\left((1-\gamma^i)V_u^i\right) = \mu_V^i du + \sigma_V^i (1-\gamma^i)V_u^i dB_u, \quad (\text{A.8})$$

where<sup>1</sup>

$$\sigma_V^i \equiv \frac{1-\gamma^i}{\gamma^i} \kappa_t - \frac{1}{\gamma^i \Xi_2^i} \frac{g^i}{g^i} \sigma_X \quad (\text{A.9})$$

$$\mu_V^i \equiv -(1-\gamma^i) f^i(c_u^i, V_u^i). \quad (\text{A.10})$$

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<sup>1</sup>Equation (A.10) follows from the definition of  $V$ , which implies

$$(1-\gamma^i) V_{t,s}^i + \int_s^t (1-\gamma^i) f^i(c_{u,s}, V_{u,s}) du = E_t \int_s^\infty (1-\gamma^i) f^i(c_{u,s}, V_{u,s}) du.$$



From the definition of  $X_t$  we obtain

$$\begin{aligned} X_t Y_t &= \int_{-\infty}^t \nu^A \pi e^{-\pi(t-s)} c_{t,s}^A ds \\ &= \int_{-\infty}^t \nu^A \pi e^{-\pi(t-s)} c_{s,s}^A e^{\frac{1}{1-\alpha^A} \int_s^t f_v^A(w) dw} \left( \frac{(1-\gamma^A)V_t^A}{(1-\gamma^A)V_s^A} \right)^{\Xi_4^A} \left( \frac{\tilde{\xi}_t}{\tilde{\xi}_s} \right)^{\frac{1}{\alpha^A-1}} ds \end{aligned} \quad (\text{A.11})$$

Applying Ito's Lemma to both sides of Equation (A.11), using (A.8), equating the diffusion and drift components on the left- and right-hand side, and simplifying gives

$$\frac{\sigma_X}{X_t} + \sigma_Y = \Xi_4^A \sigma_V^i - \frac{\kappa_t}{\alpha^A - 1} \quad (\text{A.12})$$

$$\mu_X + X_t \mu_Y + \sigma_X \sigma_Y = \nu^A \pi \beta_t^A - \pi X_t + \frac{r_t - \rho}{1 - \alpha^A} X_t + n^A X_t, \quad (\text{A.13})$$

where

$$n^i \equiv \left( \frac{q^i(q^i - 1)}{2} \kappa_t^2 + \frac{\Xi_4^i(\Xi_4^i - 1)}{2} (\sigma_V^i)^2 - q^i \Xi_4^A \kappa_t \sigma_V^i \right), \quad (\text{A.14})$$

and  $q^i \equiv \frac{1}{\alpha^i - 1}$ , for  $i \in \{A, B\}$ . Similarly, applying Ito's Lemma to both sides of the good-market clearing equation

$$Y_t = \int_{-\infty}^t \nu^A \pi e^{-\pi(t-s)} c_{t,s}^A ds + \int_{-\infty}^t \nu^B \pi e^{-\pi(t-s)} c_{t,s}^B ds, \quad (\text{A.15})$$

using (A.8), equating the diffusion and drift components on the left- and right-hand sides and combining with (A.12) and (A.13) leads to Equations (11) and (12). Using Equation (11) inside (A.12) gives (9), while (10) follows from (A.13). Finally, Equation (A.14) leads to

$$\begin{aligned} n^i(X_t) &= \frac{2 - \alpha^i}{2\gamma^i(1 - \alpha^i)} \kappa^2(X_t) + \frac{\alpha^i + \gamma^i - 1}{2\gamma^i \alpha^i} \left( \frac{g^{iu}}{g^i} \sigma_X(X_t) \right)^2 \\ &\quad - \frac{\gamma^i - \alpha^i(1 - \gamma^i)}{\gamma^i(1 - \gamma^i)} \frac{\alpha^i + \gamma^i - 1}{(1 - \alpha^i) \alpha^i} \left( \frac{g^{iu}}{g^i} \sigma_X(X_t) \right) \kappa(X_t). \end{aligned} \quad (\text{A.16})$$

The remainder of the proof shows that  $\beta_{t,s}^i$  and  $g_{t,s}^i$  are indeed functions of  $X_t$  and shows how to obtain these functions after solving appropriate ordinary differential equations. To this end, we use the parametric specification  $G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$ , and define

$$\phi^j(X_t) \equiv B_j \omega E_t \int_t^\infty e^{-(\pi + \delta_j)(u-t)} \frac{\xi_u Y_u}{\xi_t Y_t} du, \quad (\text{A.17})$$

so that an agent's net present value of earnings at birth is given by  $Y_s \phi(X_t)$  with  $\phi(x) = \phi^1(x) + \phi^2(x)$ . Next we note that equation (A.17) implies

$$e^{-(\pi + \delta_j)t} Y_t \xi_t \phi^j(X_t) + B_j \omega \int_s^t e^{-(\pi + \delta_j)u} \xi_u Y_u du = B_j \omega E_t \int_s^\infty e^{-(\pi + \delta_j)u} \xi_u Y_u du. \quad (\text{A.18})$$

Applying Ito's Lemma to both sides of equation (A.18), setting the drifts equal to each other, and using the fact that the right hand side of the above equation is a local martingale with respect to  $t$  (so that its drift is equal to zero) results in the differential equation

$$0 = \frac{\sigma_X^2}{2} \frac{d^2 \phi^j}{dX^2} + \frac{d\phi^j}{dX} (\mu_X + \sigma_X(\sigma_Y - \kappa)) + \phi^j (\mu_Y - r - \pi - \delta_j - \sigma_Y \kappa) + B_j \omega \quad (\text{A.19})$$

for  $j = 1, 2$ . A similar reasoning allows us to obtain an expression for  $g^i(X_t)$ ,  $i \in \{A, B\}$ . Since at each point in time an agent's present value of consumption has to equal her total wealth, we obtain

$$\frac{1}{g^i(X_t)} = \frac{\widetilde{W}_{t,s}^i}{c_{t,s}^i} = E_t \int_t^\infty \frac{\widetilde{\xi}_u}{\widetilde{\xi}_t} \frac{c_u^i}{c_t^i} du = E_t \int_t^\infty \frac{\widetilde{\xi}_u}{\widetilde{\xi}_t} \frac{c_u^i}{c_t^i} du. \quad (\text{A.20})$$

Using (A.3) and (A.5) gives

$$\frac{c_u^i}{c_t^i} = e^{\frac{1}{\gamma^i} \int_t^u (\Xi_1^i g^i(X_w) + \Xi_3^i) dw} \left( \frac{g^i(X_u)}{g^i(X_t)} \right)^{\frac{-\Xi_1^i}{\gamma^i}} \left( \frac{\widetilde{\xi}_u}{\widetilde{\xi}_t} \right)^{-\frac{1}{\gamma^i}}. \quad (\text{A.21})$$

Using (A.21) inside (A.20) and re-arranging implies that

$$g^i(X_t)^{-1 - \frac{\Xi_1^i}{\gamma^i}} \left( \frac{\widetilde{\xi}_t}{\widetilde{\xi}_t} \right)^{\frac{\gamma^i - 1}{\gamma^i}} e^{\frac{1}{\gamma^i} \int_0^t (\Xi_1^i g^i(X_w) + \Xi_3^i) dw} + \int_s^t \left( \frac{\widetilde{\xi}_u}{\widetilde{\xi}_t} \right)^{\frac{\gamma^i - 1}{\gamma^i}} e^{\frac{1}{\gamma^i} \int_s^u (\Xi_1^i g^i(X_w) + \Xi_3^i) dw} g^i(X_u)^{-\frac{\Xi_1^i}{\gamma^i}} du$$

is a local martingale. Applying Ito's Lemma and setting the drift of the resulting expression equal to zero gives

$$0 = \frac{\sigma_X^2}{2} M_1^i \left( (M_1^i - 1) \left( \frac{dg^i}{g^i} \right)^2 + \frac{d^2 g^i}{g^i} \right) + M_1^i \frac{dg^i}{g^i} (\mu_X - M_2^i \sigma_X \kappa) + \left( \frac{\kappa^2(X_t)}{2} M_2^i (M_2^i - 1) - M_2^i (r(X_t) + \pi) - M_1^i g^i + \frac{\Xi_3^i}{\gamma^i} \right), \quad (\text{A.22})$$

where  $M_1^i = -1 - \frac{\Xi_1^i}{\gamma^i}$  and  $M_2^i = \frac{\gamma^i - 1}{\gamma^i}$ .

Concerning boundary conditions for the ODEs (A.19) and (A.22), we note that at the boundaries  $X_t = 0$  and  $X_t = 1$  the value of  $\sigma_X(X_t)$  is zero. Accordingly, we require that all terms in equations (A.19) and (A.22) that contain  $\sigma_X$  as a multiplicative factor vanish.<sup>2</sup> We are consequently left with the following conditions when  $X_t = 0$  or  $X_t = 1$ :

$$0 = \frac{d\phi^j}{dX} \mu_X + \phi^j (\mu_Y - r - \pi - \delta_j - \sigma_Y \kappa) + B_j \omega, \quad (\text{A.23})$$

$$0 = M_1^i \frac{1}{g^i} \frac{dg^i}{dX_t} \mu_X + \left( \frac{\kappa^2(X_t)}{2} M_2^i (M_2^i - 1) - M_2^i (r(X_t) + \pi) - M_1^i g^i + \frac{\Xi_3^i}{\gamma^i} \right). \quad (\text{A.24})$$

---

<sup>2</sup>Specifically, we require that the terms  $\sigma_X^2 \frac{d^2 \phi^j}{dX^2}$ ,  $\sigma_X^2 \frac{d^2 g^i}{dX^2}$ ,  $\sigma_X \frac{d\phi^j}{dX}$ , and  $\sigma_X \frac{dg^i}{dX}$  tend to zero as  $X_t$  tends to either zero or one. These conditions ensure that any economic effects at these boundaries (on the consumption-to-wealth ratio, the present-value-of-income-to-current-output ratio, etc.) of the type of agents with zero consumption weight at the boundary occurs *only* through the anticipated (deterministic) births of new generations of such agents.

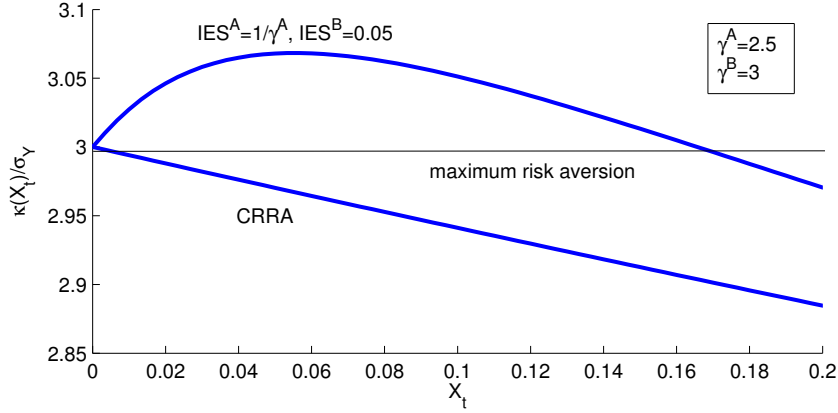


Figure A.1: An illustration of Propositions 2 and 3. The ratio  $\frac{\kappa(X_t)}{\sigma_Y}$  may be higher than the risk aversion of even the most risk-averse agent.

We conclude the proof by noting that since both  $g_{s,t}^i$  for  $i \in \{A, B\}$  and  $\phi^j$  for  $j = 1, 2$  are functions of  $X_t$ , so is  $\beta^i(X_t)$ , which is by definition equal to  $\beta^i(X_t) = g^i(X_t) \left[ \sum_{j=1}^2 \phi^j(X_s) \right]$ . The fact that  $g_{s,t}^i$  and  $\beta_t^i$  are functions of  $X_t$  verifies the conjecture that  $X_t$  is Markovian and that  $r_t$  and  $\kappa_t$  are functions of  $X_t$ , which implies that the value functions of agents  $i \in A, B$  take the form (A.4). ■

**Remark 1** We note that equation (11) implies immediately that  $\frac{\kappa(X_t)}{\sigma_Y}$  no longer equals the simple consumption-weighted (harmonic) average of individual risk aversions. In fact, as Figure A.1 illustrates,  $\frac{\kappa(X_t)}{\sigma_Y}$  can exceed the risk aversion of every agent in the economy.<sup>3</sup> In this example, a researcher using a standard, expected-utility-maximizing framework to infer the risk aversion of the representative agent would obtain an estimate exceeding the maximum risk aversion in the economy.

The following result provides an expression for the ratio of the stock price to output.

**Lemma 1** *The price-to-output ratio is given by*

$$s(X_t) \equiv \frac{S_t}{Y_t} = \left[ \frac{X_t}{g^A(X_t)} + \frac{1-X_t}{g^B(X_t)} \right] - \sum_{j=1}^2 \frac{\pi}{\pi + \delta_j} \phi^j(X_t), \quad (\text{A.25})$$

while the volatility of returns is given by

$$\sigma_t = \frac{s'(X_t)}{s(X_t)} \sigma_X(X_t) + \sigma_Y. \quad (\text{A.26})$$

<sup>3</sup>The reason behind this outcome is captured by Proposition 3, which we discuss further in Remark 3 below.

**Proof of Lemma 1.** Using (5), applying Ito's lemma to compute  $d(e^{-\pi s} \xi_s W_s^i)$ , integrating, and using a transversality condition we obtain

$$W_{t,s}^i = E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} (c_{u,s}^i - y_{u,s}) du. \quad (\text{A.27})$$

The market-clearing equations for stocks and bonds imply

$$S_t = \sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i W_{t,s}^i ds. \quad (\text{A.28})$$

Substitution of (A.27) into (A.28) gives

$$\begin{aligned} S_t &= \sum_{i \in \{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds \\ &\quad - \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,s} du \right] ds. \end{aligned} \quad (\text{A.29})$$

We can compute the first term in (A.29) as

$$\begin{aligned} &\sum_{i \in \{A,B\}} v^i \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds \\ &= \sum_{i \in \{A,B\}} v^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^i \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \frac{c_{u,s}^i}{c_{t,s}^i} du \right] ds \\ &= \sum_{i \in \{A,B\}} v^i \int_{-\infty}^t \pi e^{-\pi(t-s)} \frac{c_{t,s}^i}{g(X_t)} ds = Y_t \left[ \frac{X_t}{g^A(X_t)} + \frac{1 - X_t}{g^B(X_t)} \right] \end{aligned} \quad (\text{A.30})$$

Similarly, using the diffusion equation for output and (3) we can compute the second term in (A.29) as

$$\begin{aligned} &\int_{-\infty}^t \pi e^{-\pi(t-s)} \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,s} du \right] ds \\ &= \omega \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[ E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} Y_u \left( \sum_{j=1}^2 B_j e^{-\delta_j(u-s)} \right) du \right] ds \\ &= Y_t \times \sum_{j=1}^2 \int_{-\infty}^t \pi e^{-(\pi+\delta_j)(t-s)} \left[ B_j \omega E_t \int_t^\infty e^{-(\pi+\delta_j)(u-t)} \frac{\xi_u}{\xi_t} \frac{Y_u}{Y_t} du \right] ds \\ &= Y_t \times \sum_{j=1}^2 \int_{-\infty}^t \pi e^{-(\pi+\delta_j)(t-s)} \phi^j(X_t) ds \\ &= Y_t \times \sum_{j=1}^2 \frac{\pi}{\pi + \delta_j} \phi^j(X_t). \end{aligned} \quad (\text{A.31})$$

Combining (A.30) and (A.31) inside (A.29) gives (A.25). Equation (A.26) follows upon applying Ito's Lemma to  $S_t$  and matching diffusion terms. ■

**Proofs of Corollary 1 and Proposition 1.** Corollary 1 is a special case of Proposition 2 given by  $\gamma^A = \gamma^B$ .<sup>4</sup> Since  $\sigma_X(X_t) = 0$  when  $\gamma^A = \gamma^B$ , Lemma 1 implies that  $\sigma_t = \sigma_Y$  when  $\gamma^A = \gamma^B$ . Turning to the proof of Proposition 1, the fact that agents have the same preferences implies that  $\mu_X = \sigma_X = 0$  for all  $X_t$ ,  $g^i(X_t)$  and  $\phi^j(X_t)$  for  $i \in \{A, B\}$ ,  $j = 1, 2$  are constants, and hence equation (12) becomes (6), while (11) becomes  $\kappa = \gamma\sigma_Y^2$ . Furthermore equation (A.22) becomes an algebraic equation with solution

$$g = \pi + \frac{\rho}{1-\alpha} - \frac{\alpha}{1-\alpha} \left( r + \frac{\gamma}{2}\sigma_Y^2 \right), \quad (\text{A.32})$$

while the present value of an agent's earnings at birth, divided by  $Y_s$ , is equal to

$$\omega E_s \int_s^\infty G(u-s) \frac{\xi_u Y_u}{\xi_s Y_s} du = \omega \left( \int_0^\infty G(u) e^{-(r+\pi+\gamma\sigma_Y^2-\mu_Y)u} du \right). \quad (\text{A.33})$$

Combining (A.32) with (A.33) and the definition of  $\beta(r)$  leads to (7). ■

The following result provides sufficient conditions under which the OLG structure results in a lower interest rate than in the corresponding infinitely-lived-agent economy.

**Lemma 2** *Assume that all agents have the same preferences. Let  $\chi \equiv \rho + \pi - \alpha(\mu_Y - \frac{\gamma}{2}\sigma^2)$  and assume that  $\chi > \pi$ ,  $\mu_Y - \gamma\frac{\sigma_Y^2}{2} > 0$ , and*

$$\frac{1}{\omega} < \frac{\chi \int_0^\infty G(u) e^{-\chi u} du}{\pi \int_0^\infty G(u) e^{-\pi u} du}. \quad (\text{A.34})$$

*Then  $\beta > 1$  and hence the interest rate given by (6) is lower than the respective interest rate in an economy featuring an infinitely-lived representative agent.*

**Proof of Lemma 2.** Let  $\bar{r} \equiv \rho + (1-\alpha)\mu_Y - \gamma(2-\alpha)\frac{\sigma_Y^2}{2}$  denote the interest rate in the economy featuring an infinitely lived agent and also let  $r^* \equiv \mu_Y - \gamma\sigma_Y^2$ . Note that  $\chi > \pi$  is equivalent to  $\bar{r} > r^*$ . We first show that condition (A.34) and  $\chi > \alpha\pi$  imply, respectively, the two inequalities

$$0 > \rho + (1-\alpha)[\mu_Y + \pi(1-\beta(\bar{r}))] - \gamma(2-\alpha)\frac{\sigma_Y^2}{2} - \bar{r} \quad (\text{A.35})$$

$$0 < \rho + (1-\alpha)[\mu_Y + \pi(1-\beta(r^*))] - \gamma(2-\alpha)\frac{\sigma_Y^2}{2} - r^*. \quad (\text{A.36})$$

Since preferences are homogeneous, equation (A.32) holds. Using the definition of  $\bar{r}$  and  $\beta(\bar{r})$  on the right hand side of (A.35) and simplifying gives  $0 > 1 - \chi\omega \left( \int_0^\infty G(u) e^{-\chi u} du \right)$ ,

---

<sup>4</sup>Assuming existence of an equilibrium, the existence of a steady state follows from  $\sigma_X = 0$  for all  $X_t$ , along with  $\mu_X(0) = \pi\nu^A\beta^A(0) > 0$ ,  $\mu_X(1) = -\pi\nu^B\beta^B(1) < 0$ , and the intermediate value theorem.

which is implied by (A.34). Similarly, using the definition of  $r^*$  and  $\beta(r^*)$  inside (A.36) and simplifying gives  $0 < \left[ \rho + \pi - \alpha(\mu_Y + \pi - \gamma \frac{\sigma_Y^2}{2}) \right] (1 - \omega)$ , which is implied  $\chi > \alpha\pi$ . Given the inequalities (A.35) and (A.36), the intermediate value theorem implies that there exists a root of equation (6) in the interval  $(r^*, \bar{r})$ . Accordingly,  $\beta > 1$ . ■

**Remark 2** *In an influential paper, Weil (1989) pointed out that the standard representative-agent model cannot account for the low level of the risk-free rate observed in the data. Motivated by the low estimates of the IES in Hall (1988) and Campbell and Mankiw (1989), Weil’s reasoning was that such values of the IES would lead to interest rates that are substantially higher than the ones observed in the data. Weil referred to this observation as the “low risk-free rate puzzle”. Lemma 2 gives sufficient conditions so that the interest rate in our OLG economy is lower than in the respective infinitely-lived, representative-agent economy. Whether the key condition (A.34) of the Lemma holds or not depends crucially on the life-cycle path of earnings. The easiest way to see this is to assume that  $\omega\chi > \pi$  and restrict attention to the parametric case  $G(u) = e^{-\delta u}$ , so that condition (A.34) simplifies to  $\delta > \frac{\chi\pi(1-\omega)}{\omega\chi-\pi}$ . Hence, the interest rate is lower in the overlapping generations economy as long as the life cycle path of earnings is sufficiently downward-sloping. The intuition for this finding is that agents who face a downward-sloping path of labor income need to save for the latter years of their lives. The resulting increased supply of savings lowers the interest rate. This insight is due to Blanchard (1985), who considered only the deterministic case and exponential specifications for  $G(u)$ . Condition (A.34) generalizes the results in Blanchard (1985). In particular, it allows  $G(u)$  to have any shape, potentially even sections where the life-cycle path of earnings is increasing.*

*We illustrate the quantitative content of this observation with a numerical example. Figure A.2 depicts the quantitative magnitude of the difference in riskless rates between an economy featuring an infinitely lived agent and our OLG economy. We use the parameters chosen in Section B. For  $G(t-s)$  we use the exact path of life-cycle earnings reported in Hubbard et al. (1994), which we depict in Figure B.1. It is interesting to note, for the results obtained in the heterogeneous-preference calibrations, that interest rates remain basically unchanged when we use the parametric specification for  $G(t-s)$  of equation (3) with parameters estimated by non-linear least squares. Figure A.2 shows that, for the low levels of IES that lead to implausibly high interest rates in the economy featuring an infinitely-lived agent, the OLG counterpart generates lower interest rates.*

**Proof of Proposition 3.** We start by fixing arbitrary values  $\alpha^A$ ,  $\alpha^B$ , and  $\gamma$ , and throughout we let  $\gamma^B = \gamma$  and  $\gamma^A = \gamma^B - \varepsilon$ ,  $\varepsilon \geq 0$ . For  $\kappa(X_t)$  given by (11), we let  $Z(X_t; \varepsilon) \equiv \kappa(X_t) - \Gamma(X_t)\sigma_Y$ . Clearly  $Z(\bar{X}; 0) = 0$  (by Proposition 1). To prove the proposition, it suffices to show that  $D_\varepsilon Z(\bar{X}; 0) > 0$ , where  $D_\varepsilon Z$  denotes the derivative of  $Z$  with respect to  $\varepsilon$ . (Note that  $\bar{X}$  is also a function of  $\varepsilon$ .) Direct differentiation, along with the fact that  $\sigma_X = 0$  when  $\varepsilon = 0$ , gives

$$D_\varepsilon Z(\bar{X}; 0) = \sum_{i \in \{A, B\}} \omega^i(\bar{X}) \left( \frac{1 - \gamma^i - a^i}{\alpha^i} \right) \frac{g^i(\bar{X}; 0)}{g^i(\bar{X}; 0)} \times D_\varepsilon \sigma_X(\bar{X}; 0). \quad (\text{A.37})$$

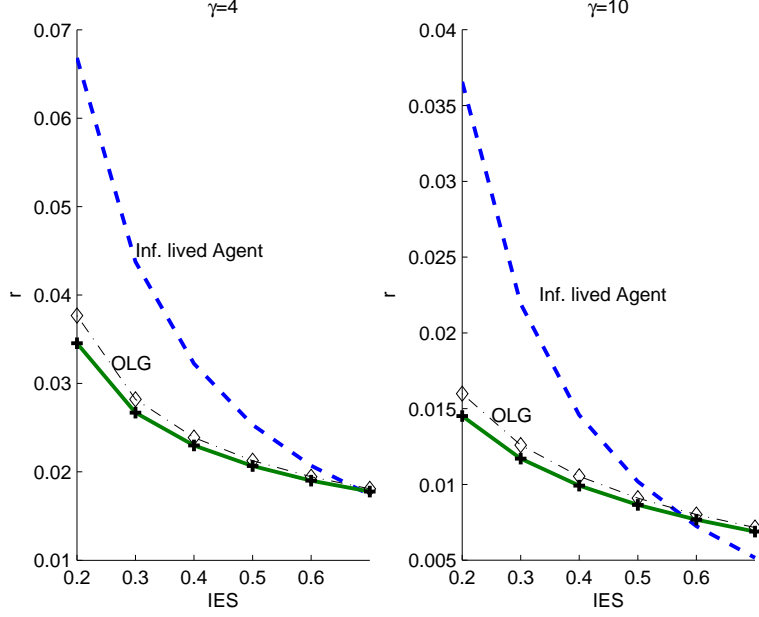


Figure A.2: Interest rates when all agents have identical preferences. The dashed line pertains to an economy populated by an infinitely-lived agent, while the other two lines to our OLG model. The continuous line obtains when using the exact path for  $G(u)$  reported in Hubbard et al. (1994) and performing the integration in equation (7) numerically. The dash-dot line obtains when using non-linear least squares to project  $G(\cdot)$  on a sum of scaled exponential functions, and then calculating the integral in equation (7) exactly. For the computations, we set a positive, but small, value of  $\rho = 0.001$  to illustrate that the high risk free rates (in the case where agents are infinitely lived) are not the result of a high discount rate. We set  $\mu_Y = 0.02$  and  $\sigma_Y = 0.041$  to match the mean and the volatility of the yearly, time-integrated consumption growth rate,  $\pi = 0.02$  to match the birth rate, and  $\omega = 0.92$  to match the share of labor income in national income. Details on the sources of the data and further discussion of these parameters are contained in Section B).

To determine the sign of  $\frac{g^{i'}(\bar{X};0)}{g^i(\bar{X};0)}$ , we write equation (A.22) — for  $\varepsilon = 0$  — as

$$0 = M_1^i \frac{g^{i'}}{g^i} \mu_X + \left[ \frac{(\gamma\sigma)^2}{2} M_2^i (M_2^i - 1) - M_2^i (r(X_t) + \pi) + \frac{\Xi_3^i}{\gamma} - M_1^i g^i \right]. \quad (\text{A.38})$$

Differentiating (A.38) with respect to  $X$  and evaluating the result at  $X = \bar{X}$  using  $\mu_X = 0$  yields

$$\frac{g^{i'}}{g^i} M_1^i (\mu'_X - g^i) = M_2^i r', \quad (\text{A.39})$$



where primes denote derivatives with respect to  $X$ . Re-arranging (A.39) and using the definitions of  $M_1^i$  and  $M_2^i$  gives

$$\frac{g^{i'}}{g^i} = \frac{M_2^i}{M_1^i} \left( \frac{r'}{\mu'_X - g^i} \right) = \frac{\alpha^i}{1 - \alpha^i} \frac{r'}{\mu'_X - g^i}. \quad (\text{A.40})$$

A similar computation starting from equation (A.19) and using  $\phi^j(\bar{X}) = \frac{B_j \omega}{r + \pi + \delta_j + \gamma \sigma^2 - \mu_Y}$  yields

$$\frac{\phi^{j'}}{\phi^j} = - \frac{1}{\frac{B_j \omega}{\phi^j} - \mu'_X} r'. \quad (\text{A.41})$$

The fact that  $\bar{X}$  is a stable steady state<sup>5</sup> implies that  $\mu'_X \leq 0$ , and accordingly equation (A.40) implies that  $\frac{1}{\alpha} \frac{g^{i'}}{g^i}$  has the opposite sign from  $r'$ . Similarly, equation (A.41) implies that  $\frac{\phi_j'}{\phi_j}$  has the opposite sign<sup>6</sup> from  $r'$ .

We next show that if  $\gamma^A = \gamma^B = \gamma$  and  $\alpha^B > \alpha^A$ , then  $r'(\bar{X}) > 0$ . We proceed by supposing the contrary:  $\gamma^A = \gamma^B = \gamma$  and  $\alpha^B > \alpha^A$ , but  $r' \leq 0$ . Differentiating equation (12) with respect to  $X_t$  and evaluating the resulting expression at  $X_t = \bar{X}$ , we obtain

$$r' = \frac{1}{\Theta(\bar{X})} \left\{ \left( \frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A} \right) [r - \rho] - n^A + n^B \right\} - \pi \frac{1}{\Theta(\bar{X})} \sum_i v^i (\beta^i)'. \quad (\text{A.42})$$

The definition of  $n^i(X_t)$  in equation (A.16) along with  $\gamma^A = \gamma^B = \gamma$  and  $\sigma_X = 0$  gives

$$n^A - n^B = \left[ \frac{2 - \alpha^A}{(1 - \alpha^A)} - \frac{2 - \alpha^B}{(1 - \alpha^B)} \right] \frac{\kappa^2(\bar{X})}{2\gamma} = \gamma \left( \frac{1}{1 - \alpha^A} - \frac{1}{1 - \alpha^B} \right) \frac{\sigma_Y^2}{2}. \quad (\text{A.43})$$

Furthermore, noting that  $\beta^i = g^i \sum_{j=1,2} \phi^j$  and using (A.40) and (A.41) leads to

$$\begin{aligned} \pi \sum_{i \in \{A,B\}} v^i (\beta^i)' &= -r' \left[ \pi \sum_{i \in \{A,B\}} v^i \left( \frac{\alpha^i}{1 - \alpha^i} \frac{1}{g^i - \mu'_X} g^i \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \phi_j \frac{1}{\frac{B_j \omega}{\phi^j} - \mu'_X} \right) \right] \\ &\leq -r' \left[ \pi \sum_{i \in \{A,B\}} v^i \left( \frac{(\alpha^i)^+}{1 - \alpha^i} \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \frac{\phi_j^2}{B_j \omega} \right) \right]. \end{aligned} \quad (\text{A.44})$$

Inequality (A.44) follows from the facts that (a)  $r'$  has been assumed to be non-positive, (b)  $\mu'_X \leq 0$ , and (c)  $\sum_{j=1,2} \phi_j \frac{1}{\frac{B_j \omega}{\phi^j} - \mu'_X}$  is positive and increasing in  $\mu'_X$ .<sup>7</sup>

<sup>5</sup>Note that when  $\varepsilon = 0$ , the evolution of  $X_t$  is deterministic.

<sup>6</sup>Note that  $B_j$  and  $\phi^j$  have the same sign.

<sup>7</sup>To see that  $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega - \mu'_X}$  is positive, note that  $\frac{1}{(\phi^1)^{-1} B_1 \omega - \mu'_X} = \frac{1}{r + \pi + \delta_1 + \sigma_Y \kappa - \mu_Y - \mu'_X} > \frac{1}{r + \pi + \delta_2 + \sigma_Y \kappa - \mu_Y - \mu'_X}$  since  $\delta_1 < \delta_2$ . Furthermore, since  $B_1 > -B_2$ , it follows that  $\phi^1 > -\phi^2$ . Treating  $\mu'_X$  as a variable and differentiating  $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega - \mu'_X}$  with respect to  $\mu'_X$  shows that  $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega - \mu'_X}$  is increasing in  $\mu'_X$ .

Let

$$\eta \equiv \pi \sum_{i \in \{A, B\}} v^i \left( \frac{(\alpha^i)^+}{1 - \alpha^i} \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \frac{\phi_j^2}{B_j \omega} \right). \quad (\text{A.45})$$

Let  $\alpha^A = \alpha^B = \alpha$ . Using  $\phi^j(\bar{X}) = \frac{B_j \omega}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y}$  and equation (A.32), we obtain that

$$\eta = \pi \left( \frac{\alpha^+}{1 - \alpha} \sum_{j=1,2} \frac{\omega B_j}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y} + \sum_{j=1,2} \frac{\omega B_j \left( \pi + \frac{\rho}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \left( r + \frac{\gamma}{2} \sigma_Y^2 \right) \right)}{\left( r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y \right)^2} \right),$$

where  $r$  is given in equation (6). We shall assume that

$$\eta < 1 - \alpha, \quad (\text{A.46})$$

which is the case, for instance, when  $\omega$  is sufficiently small.<sup>8</sup> Using (A.44) and (A.43) inside (A.42) gives

$$r' \geq \frac{1}{\Theta(\bar{X})} \left( \frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A} \right) \left( r - \rho + \gamma \frac{\sigma_Y^2}{2} \right) + \frac{\eta}{\Theta(\bar{X})} r', \quad (\text{A.47})$$

or

$$(\Theta(\bar{X}) - \eta) r' \geq \left( \frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A} \right) \left( r - \rho + \gamma \frac{\sigma_Y^2}{2} \right). \quad (\text{A.48})$$

The term  $r - \rho + \gamma \frac{\sigma_Y^2}{2}$  is positive if<sup>9</sup>

$$\mu_Y - \frac{\gamma \sigma_Y^2}{2} + \pi > \omega (\rho + \pi) \frac{\sum_{j=1,2} \frac{B_j}{\delta_j + \rho - \mu + \frac{\gamma}{2} \sigma_Y^2 + \pi}}{\sum_{j=1,2} \frac{B_j}{\delta_j + \pi}}, \quad (\text{A.49a})$$

$$\rho + \pi (1 - \alpha) > \alpha \left( \mu_Y - \gamma \frac{\sigma_Y^2}{2} \right). \quad (\text{A.49b})$$

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<sup>8</sup>Note that an implication of Lemma 2 is that the interest rate satisfies  $r > \mu_Y - \gamma \sigma_Y^2$  for any value of  $\omega$ , and hence the expressions  $\frac{1}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y}$  and  $\frac{\pi + \frac{\rho}{1 - \alpha^i} - \frac{\alpha^i}{1 - \alpha^i} \left( r + \frac{\gamma}{2} \sigma_Y^2 \right)}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y}$  must approach finite limits as  $\omega$  goes to zero.

<sup>9</sup>When the economy is populated by a single agent, we can replicate the steps of the proof of Lemma 2 to prove this fact. Specifically, letting  $r^{**} = \rho - \frac{\gamma}{2} \sigma_Y^2$  we arrive at the conclusion that

$$0 < \rho + (1 - \alpha) [\mu_Y + \pi (1 - \beta(r^{**}))] - \gamma (2 - \alpha) \frac{\sigma_Y^2}{2} - r^{**},$$

as long as condition (A.49a) holds. Similarly, setting  $\bar{r} = \rho + (1 - \alpha) (\mu_Y + \pi) - \gamma (2 - \alpha) \frac{\sigma_Y^2}{2}$  gives

$$0 > \rho + (1 - \alpha) [\mu_Y + \pi (1 - \beta(\bar{r}))] - \gamma (2 - \alpha) \frac{\sigma_Y^2}{2} - \bar{r}.$$

as long as equation (A.49b) holds. Hence there must exist a root in the interval  $(r^{**}, \bar{r})$ .

We note that condition (A.49b) is automatically satisfied when  $\alpha < 0$  and  $\mu_Y \geq \gamma \frac{\sigma_Y^2}{2}$ , while condition (A.49a) holds when  $\omega$  is small.

When  $|\alpha^A - \alpha^B|$  is small (but not zero), continuity implies that the right hand side of (A.48) has the same sign as  $\frac{1}{1-\alpha^B} - \frac{1}{1-\alpha^A}$ , which is positive. However, given the supposition that  $r'$  is non-positive and assumption (A.46), the left-hand side of (A.48) is non-positive, which is a contradiction. We therefore conclude that, when  $\alpha^B > \alpha^A$ ,  $r' > 0$ . A symmetric argument implies that, when  $\alpha^B < \alpha^A$ ,  $r' < 0$ . Recalling that  $r'$  and  $\frac{1}{\alpha^i} \frac{g^{i'}}{g^i}$  have the same sign, it follows that the term  $\sum_{i \in \{A, B\}} \omega^i(\bar{X}) \left( \frac{1-\gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^{i'}(\bar{X}; 0)}{g^i(\bar{X}; 0)}$  is positive when either condition 1 or condition 2 of the Lemma holds.

Consider now the term  $D_\varepsilon \sigma_X(\bar{X}; 0)$ , which we want positive. Direct differentiation of (9) gives

$$D_\varepsilon \sigma_X(\bar{X}; 0) = \frac{\bar{X} (D_\varepsilon \Gamma(\bar{X}; 0) + 1)}{\bar{X} (1 - \bar{X}) \left[ \frac{1-\gamma-\alpha^A}{\alpha^A} \frac{g^{A'}}{g^A} - \frac{1-\gamma-\alpha^B}{\alpha^B} \frac{g^{B'}}{g^B} \right] + \gamma} \sigma_Y. \quad (\text{A.50})$$

The numerator on the right hand side of equation (A.50) is positive, and as long as  $\alpha^A$  and  $\alpha^B$  are close enough,  $\frac{1-\gamma-\alpha^A}{\alpha^A} \frac{g^{A'}}{g^A} - \frac{1-\gamma-\alpha^B}{\alpha^B} \frac{g^{B'}}{g^B}$  is arbitrarily close to zero, so that the denominator is also positive.

The proof can now be concluded by invoking uniform continuity on compact sets. Specifically, the right-hand side of equation (A.48) is strictly positive, for every  $\gamma$ , if  $\alpha^B > \alpha^A$ . For  $\alpha^B - \alpha^A$  sufficiently small, the right-hand-side of (A.50) is also positive.

Thus, for fixed  $\gamma$  and  $\alpha_A$ , there exists  $D\alpha > 0$  such that  $0 < \alpha_B - \alpha_A < D\alpha$  implies  $D_\varepsilon Z(\bar{X}; 0) > 0$ . In that case, there exists  $\bar{\varepsilon} > 0$  such that  $Z(\bar{X}; \varepsilon) \geq 0 \forall \varepsilon \in [0, \bar{\varepsilon}]$ . The quantity  $D\alpha$  can be chosen continuously in  $(\gamma, \alpha^A)$ , while  $\bar{\varepsilon}$  can be chosen continuously in  $(\gamma, \alpha_A, \alpha_B)$ .

Now restrict attention to  $\gamma \in K^\gamma$  and  $\alpha^A \in K^\alpha$ , for compact sets  $K^\gamma$  and  $K^\alpha$ . It follows that  $D\alpha^* > 0$  exists so that  $D_\varepsilon Z(\bar{X}; 0) > 0$  whenever  $0 < \alpha_B - \alpha_A < D\alpha^*$ , for any  $\gamma \in K^\gamma$ . Now pick  $d\alpha > 0$  such that  $d\alpha < D\alpha^*$ . Then  $\varepsilon^*$  exists such that  $Z(\bar{X}; \varepsilon) \geq 0 \forall \varepsilon \in [0, \varepsilon^*]$  whenever  $(\gamma, \alpha^A) \in K^\gamma \times K^\alpha$  and  $d\alpha < \alpha^B - \alpha^A < D\alpha^*$ . ■

**Remark 3** *As we remarked in the text, the economic argument for Proposition 3 is based on the interaction between individual agents' consumption-growth autocorrelation and preferences for early resolution of uncertainty, which lies behind the long-run-risk approach to asset pricing pioneered by Bansal and Yaron (2004). Whether agents have preferences for early resolution of uncertainty is a simple parametric condition —  $\gamma^i + \alpha^i > 1$  or not — but whether individual consumption reacts to shocks more in the long run than in the short run is more subtle. We present the intuition behind this fact using a graphical argument.*

*We consider case (i), illustrated in Figure A.3; case (ii) can be treated symmetrically. The solid lines depict the paths of expected (log) consumption for an agent of type A and an agent of type B. According to case (i) of Proposition 3, agent A has a lower IES than agent B ( $\alpha^A < \alpha^B$ ). This is reflected in that agent A's expected log consumption path in Figure A.3 is flatter than that of agent B. Now suppose that a positive aggregate-consumption shock*

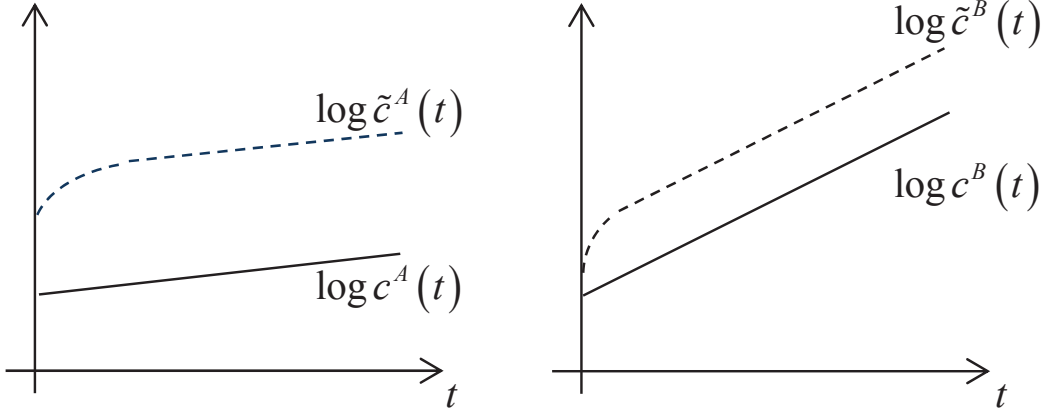


Figure A.3: Illustration of case (i) of Proposition 3.

arrives. The dotted lines describe the reaction of agents' log consumption in response to that shock. Since agent A is less risk averse, her consumption is more strongly exposed to aggregate risk. This is reflected in the larger vertical distance between the solid and the dotted line for agent A as compared to agent B; in other words,  $X$  increases.

Furthermore, the agents whose economic importance increases following the shock (type-A agents) are also the ones with the lower IES, according to the assumptions of case (i). If type-A agents kept enjoying the same relatively flat consumption growth path going forward, consumption-market clearing would require an excessively steep path for type-B agents, which they are reluctant to accept at the old prices. Equilibrium asset prices consequently have to adjust to incentivize agents to accept a steeper consumption path, and the expected consumption growth of both types of agents would increase. This increase in agents' consumption growth is temporary; it dissipates in the long run as  $X_t$  returns to its long-run mean. Thus, the immediate response in either agent's log consumption is smaller than the response in the long run.

**Proof of Proposition 4.** We start by letting  $\tilde{\sigma}_X(X_t) \equiv \frac{\sigma_X(X_t)}{\sigma_Y}$  and  $\tilde{\gamma}(X_t) = \frac{\kappa(X_t)}{\sigma_Y}$ . Expressing equations (9)–(12) in terms of  $\tilde{\sigma}_X(X_t)$  and  $\tilde{\gamma}(X_t)$  (rather than  $\sigma_X(X_t)$  and  $\kappa(X_t)$ ) and inspecting the resulting equations shows that there are no first-order  $\sigma_Y$  terms in the equations for  $r(X_t)$  and  $\tilde{\gamma}(X_t)$ . Rather, the lowest (non-zero) order is two. A similar observation applies for the differential equations (A.19) and (A.22) for  $\phi^j$  and  $g^i$ , respectively.

To establish the first two equations in (13) it therefore suffices to show that there exist parameters  $\alpha^A$ ,  $\alpha^B$ ,  $\gamma^A$ ,  $\gamma^B$ , and  $v^A$  such that

$$\frac{\kappa(X_t)}{\sigma_Y} = \Gamma(X_t) + \sum_{i \in \{A, B\}} \omega^i(X_t) \left( \frac{1 - \gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^i \sigma_X(X_t)}{g^i \sigma_Y} \quad (\text{A.51})$$

is equal to  $D_1$  when  $\sigma_Y = 0$ , while  $\frac{\kappa'(X_t)}{\sigma_Y}$  is equal to  $-D_2$ . To establish these facts, we proceed in two steps. First we show that, when  $\alpha^A = \alpha^B = \alpha$  and  $\sigma_Y = 0$ , the second term

on the right-hand side of (A.51) is zero, and so is its derivative with respect to  $X_t$ . In the second step we show that there exist  $\gamma^A$ ,  $\gamma^B$ , and small enough  $v^A$  such that  $\Gamma(\bar{X}) = D_1$  and  $\Gamma'(\bar{X}) = -D_2$ . In a third step we address the requirement  $\lim_{x \rightarrow 0} \epsilon_3(x) = 0$  and its implication.

*Step 1.* Let  $\alpha^A = \alpha^B \equiv \alpha$  and  $\sigma_Y^2 = 0$ . Then  $\Theta(X)$  is constant and  $n^i(X) = 0$ , and equation (12) gives

$$r(X_t) = \rho + (1 - \alpha) \left[ \mu_Y - \pi \left( \sum_{i \in \{A, B\}} v^i \beta^i(X_t) - 1 \right) \right]. \quad (\text{A.52})$$

The consumption ratio  $\beta^i(X)$ , in turn, is given by  $\beta^i(X) = g^i(X)(\phi^1(X) + \phi^2(X))$ . Note that, if  $g^i$  and  $\phi^j$  are constant (i.e., independent of  $X_t$ ), then so is  $r$ . On the other hand, equations (A.22) and (A.19), under the standing assumption  $\sigma_Y^2 = 0$ , are satisfied by constant functions  $g^i$  and  $\phi^j$  as long as  $r$  is constant. We conclude that the system of equations (A.19), (A.22), and (A.52) admit solutions  $r$ ,  $g^i$ , and  $\phi^j$  that are independent of  $X_t$  and therefore  $g^{i'} = g^{i''} = 0$ . Accordingly, the second term on the right-hand side of (A.51) and its derivative with respect to  $X_t$  are both zero.

*Step 2.* Now, fixing  $\alpha_A = \alpha_B = \alpha$ , suppose that we want to find  $\gamma^A$  and  $\gamma^B$  to achieve

$$\Gamma(\bar{X}) = D_1 \quad (\text{A.53})$$

$$\Gamma'(\bar{X}) = - \left( \frac{1}{\gamma^A} - \frac{1}{\gamma^B} \right) \Gamma(\bar{X})^2 = -D_2. \quad (\text{A.54})$$

With  $\sigma_Y = 0$ ,  $\bar{X} = v^A$  is a constant.<sup>10</sup> We compute  $\gamma^A$  and  $\gamma^B$  explicitly by solving the following linear system in  $1/\gamma^A$  and  $1/\gamma^B$ :

$$\frac{\bar{X}}{\gamma^A} + \frac{1 - \bar{X}}{\gamma^B} = D_1^{-1} \quad (\text{A.55})$$

$$\frac{1}{\gamma^A} - \frac{1}{\gamma^B} = D_1^{-2} D_2. \quad (\text{A.56})$$

The solution is

$$\gamma^B = D_1^{-1} (1 - \bar{X} D_1^{-1} D_2) \quad (\text{A.57})$$

$$\gamma^A = D_1^{-1} + D_1^{-2} D_2 (1 - \bar{X}). \quad (\text{A.58})$$

We note that  $\gamma^B$  is automatically positive, while  $\gamma^A > 0$  as long as  $\bar{X} = v^A$  is small enough.

For future reference we note that steps 1 and 2 (adapted in a straightforward way) generate the desired results on  $\Gamma(\bar{X})$  and  $\Gamma'(\bar{X})$  not only in the case  $\alpha^A = \alpha^B$ , but more generally when  $|\alpha^A - \alpha^B| \in O(\sigma_Y^2)$ .

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<sup>10</sup>  $\bar{X} = v^A$  because, at  $\alpha^A = \alpha^B$  and  $\sigma_Y = 0$ ,  $\beta^A = \beta^B$  and thus equation (10) implies that  $\mu_X(v^A) = 0$ .

The next step addresses the variability of the interest rate.

*Step 3.* Fix now  $\alpha^B = \alpha$  at the value chosen in Step 1, let  $\Delta\alpha = \alpha^A - \alpha^B$ , and consider the Taylor expansion

$$r(X; \Delta\alpha, \sigma_Y^2) = r_0(X; \Delta\alpha) + r_1(X; \Delta\alpha)\sigma_Y^2 + o(\sigma_Y^2) \quad (\text{A.59})$$

$$r'(X; \Delta\alpha, \sigma_Y^2) = r'_0(X; \Delta\alpha) + r'_1(X; \Delta\alpha)\sigma_Y^2 + o(\sigma_Y^2). \quad (\text{A.60})$$

We want to show that there exists  $\Delta\alpha$  such that  $r'_0(\bar{X}; \Delta\alpha) + r'_1(\bar{X}; \Delta\alpha)\sigma_Y^2 = 0$ . We note that — as established in step 1 —  $r'_0(\bar{X}; 0) = 0$  and that  $\frac{\partial}{\partial \Delta\alpha} r'_0(\bar{X}; 0) \neq 0$  generically. Now, if  $r'_1(\bar{X}; 0) = 0$ , then clearly  $r'_0(\bar{X}; \Delta\alpha) + r'_1(\bar{X}; \Delta\alpha)\sigma_Y^2 = 0$ . If  $r'_1(\bar{X}; 0) \neq 0$ , the mapping  $\Delta\alpha \mapsto \frac{r'_0(\bar{X}; \Delta\alpha)}{r'_1(\bar{X}; \Delta\alpha)}$  equals zero at zero and has non-zero derivative at zero, so its range contains a neighborhood of zero. Consequently, for  $\sigma_Y^2$  small enough,  $\Delta\alpha$  exists such that  $r'_0(\bar{X}; \Delta\alpha) + r'_1(\bar{X}; \Delta\alpha)\sigma_Y^2 = 0$ .

Furthermore,  $\Delta\alpha$  is in  $O(\sigma_Y^2)$ , which means — in light of our remark at the end of step 2 — that by choosing  $\Delta\alpha$  so as to ensure that  $\epsilon_3$  is in  $o(\sigma_Y^2)$ , the remainder terms  $\epsilon_1$  and  $\epsilon_2$  remain in  $O(\sigma_Y^2)$ .<sup>11</sup>

Now the fact that for the parameter choices made above  $\frac{(\kappa(\bar{X})\sigma(\bar{X}))' |_{\sigma_X(\bar{X})}}{|r'(\bar{X})|_{\sigma_X(\bar{X})}}$  approaches infinity as  $\sigma_Y$  approaches zero is immediate in light of (13) and (A.26).<sup>12</sup> ■

## B Calibration details

For all quantitative exercises, we set  $\mu_Y = 0.02$  and  $\sigma_Y = 0.041$ , so that time-integrated data from our model can roughly reproduce the first two moments of annual consumption growth. We note that, due to time integration, a choice of instantaneous volatility  $\sigma_Y = 0.041$  corresponds to a volatility of 0.033 for model-implied, time-integrated, yearly consumption data, consistent with the long historical sample of Campbell and Cochrane (1999). The parameter  $\pi$  controls the birth-and-death rate, and we set  $\pi = 0.02$  so as to approximately match the birth rate (inclusive of the net immigration rate) in the US population. (Source: U.S. National Center for Health Statistics, Vital Statistics of the United States, and National Vital Statistics Reports (NVSR), annual data, 1950-2006.) We note that since the death and birth rate are the same in the model,  $\pi = 0.02$  implies that on average agents live for 50 years after they start making economic decisions. Assuming that this age is about 20 years in the data,  $\pi = 0.02$  implies an average lifespan of 70 years. We set  $\omega = 0.92$  to match the fact that dividend payments and net interest payments to households are  $1 - \omega = 0.08$  of personal income. (Source: Bureau of Economic Analysis, National income and product accounts, Table 2.1., annual data, 1929-2010.) To arrive at this number, we combine dividend and net interest payments by the corporate sector (i.e., we exclude net interest payments from the government and net interest payments to the rest of the world), in order to capture total

<sup>11</sup>Under common regularity conditions, the solutions of differential equations depend smoothly on the parameters.

<sup>12</sup>Specifically, the observations we made in steps 1-3 imply that for our parameter choices  $\sigma'(\bar{X}) \in o(\sigma_Y)$ .

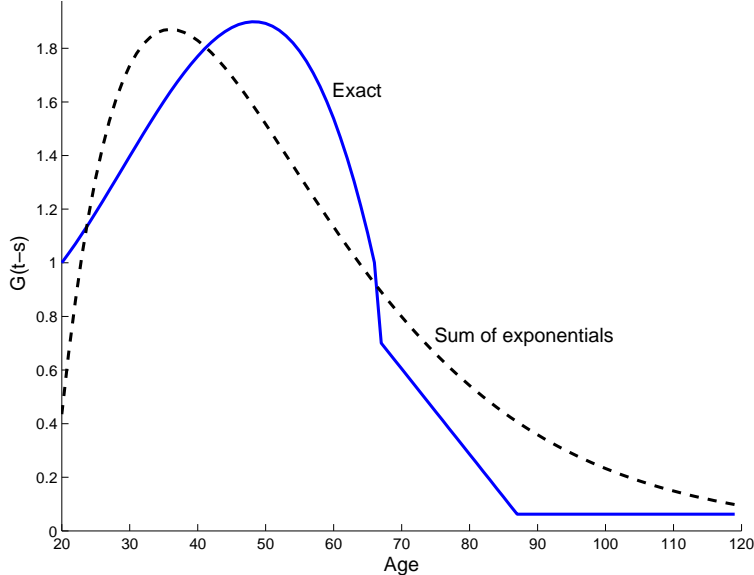


Figure B.1: Hump-shaped profile of earnings over the life-cycle. The continuous line reports the profile estimated by Hubbard et al. (1994), while the broken line depicts the non-linear least-squares projection of the earnings profile in the data on a sum of scaled exponentials  $G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$ . The estimated coefficients are  $B_1 = 30.72$ ,  $B_2 = -30.29$ ,  $\delta_1 = 0.0525$ , and  $\delta_2 = 0.0611$ .

flows from the corporate to the household sector. We note that the choice of  $1 - \omega = 0.08$  is consistent with the gross profit share of GDP being about 0.3, since the share of output accruing to capital holders is given by the gross profit share *net of the investment share*. As a result, in our endowment economy, which features no investment, it would be misleading to match the gross profit share, since this would not appropriately deduct investment, which does not constitute “income” for the households. Instead it seems appropriate to match the parameter  $\omega$  directly to the fraction of national income that accrues in the form of dividends and net interest payments from the domestic corporate sector.

For the specification of the life-cycle earnings  $G(t - s)$  we use the life-cycle profile of earnings estimated in Hubbard et al. (1994). We estimate the parameters  $B_1$ ,  $\delta_1$ ,  $B_2$ , and  $\delta_2$  using non-linear least squares to project (3) on the life-cycle profile of earnings estimated in Hubbard et al. (1994). These parameters are  $B_1 = 30.72$ ,  $B_2 = -30.29$ ,  $\delta_1 = 0.0525$ , and  $\delta_2 = 0.0611$ . Figure B.1 illustrates the fit of the parameterized function  $G$  to the empirical one.

To obtain simulated cross sections of households in Section 4, we simulate 1,000 independent paths of aggregate shocks over 4,000 years starting with  $X_0$  at its stationary mean and isolate the last 300 years of each path to ensure stationarity. Fixing each of the 1,000 paths in turn, we draw from the population age distribution the ages of 4,000 artificial households. We then repeat this exercise for  $T - k$ , where  $k$  are numbers chosen to reflect the number of years between the cross sections considered in Wolff (2010) and Cutler and Katz (1992)



Horizon (Years)	Data (Long Sample)	Model
1	-0.006	-0.013 [-0.060,0.005]
3	-0.039	-0.031 [-0.157,0.012]
5	-0.073	-0.047 [-0.248,0.019]
7	-0.116	-0.059 [-0.336,0.025]

Table C.1: Long-horizon regressions of the real interest rate on the log P/D ratio. To account for the well-documented finite-sample biases driven by the high autocorrelation of the P/D ratio, the simulated data are based on 1000 independent simulations of 106-year long samples, where the initial condition for  $X_0$  for each of these simulation paths is drawn from the stationary distribution of  $X_t$ . For each of these 106-year long simulated samples, we run predictive regressions of the form  $\log R_{t \rightarrow t+h}^f = \alpha + \beta \log(P_t/D_t)$ , where  $R_t^f$  is the gross riskless rate of return at time  $t$  and  $h$  is the horizon for returns in years. We report the median values for the coefficient  $\beta$  of these regressions, along with the respective [0.025, 0.975] confidence intervals.

respectively. As a result, we obtain 1,000 independent, simulated versions with the same cross-sectional and time-series characteristics as Wolff (2010) and Cutler and Katz (1992). Since different households experience different shocks over their life-cycle, the cross-sectional consumption and wealth distributions are continuous (and time varying) in the model. As in Wolff (2010) and Cutler and Katz (1992), we choose the Gini coefficient to summarize inequality and the volatility of yearly changes to the Gini coefficient to summarize variations in inequality.

## C Additional quantitative results

### C.1 Conditional moments: interest rate predictability

As we note in the body of the paper, interest rate movements in our model are of much smaller magnitude than excess-return movements, and consequently the price-dividend ratio predicts excess returns quite well.

Table C.1 reinforces this point. In this table we use the current log-price-dividend ratio to predict real riskless returns. Similar to the data, the model produces negative and very small coefficients of predictability of the riskless rate, while the respective coefficients for excess returns (Table 2) are an order of magnitude larger. (It is useful to note here that the negative comovement between the real interest rate and the log price-to-dividend ratio should not be interpreted as saying that the real interest rate is “countercyclical”. In our model all shocks are permanent and hence there is no notion of a “cycle” in the conventional,

Beveridge-Nelson sense nor a notion of an “output gap”.) Finally, we also note that in Table C.1 we don’t include comparisons between the  $R^2$  of regressions in the data and the model. The reason is that, unlike regression coefficients, the  $R^2$  depends on inflation shocks, which we do not model.

## C.2 Pathwise comparison with the data

We conclude with a figure showing that the model can roughly reproduce even more detailed, pathwise properties of excess returns. Specifically, we perform the following exercise. We take annual consumption data since 1889. We pick the starting value  $X_0$  (where year zero is 1889) to equal the model-implied stationary mean of  $X_t$ . For the realized consumption data between 1889 and 2010, we construct the model-implied path of  $X_t$ . Since  $X_t$  is the only state variable in the model, we can use knowledge of  $X_t$  to construct a series of model-implied excess returns  $\mu_t - r_t$  for  $t = 1889, 1890$ , and so on. The top panel of Figure C.1 compares the model-implied excess returns at time  $t$  with the realized excess return in the data. The bottom panel performs the same exercise averaging returns over rolling eight-year periods.

Clearly, the model fails to reproduce the data exactly both at the one-year and the eight-year horizon. This is hardly surprising given the stylized nature of the model. Of particular interest for our purposes is the behavior of the model for excess returns at eight-year horizons, since averaging helps isolate low-frequency movements in excess returns, which are more strongly influenced by movements in expected excess returns rather than shocks.<sup>13</sup> Focusing on the bottom panel of the figure, we note that the model performs reasonably well until 1945 and then again from the 1970’s onwards. The model fails to capture the behavior of average returns between 1945 and 1970. Interestingly, in the data the rolling eight-year excess return peaks in 1948 and then progressively declines until 1966. In the model excess returns remain around their long-term average until 1953, while the rapid consumption gains of the 1950s and 1960s lead to a run-up in excess returns that reaches a peak in 1960, followed by a decline. One potential explanation for the failure of the model during that period is that some of the productivity and consumption gains that followed the war were not shocks, but rather gains anticipated by market participants who understood that productivity “catches up” rapidly after a war. Accordingly, the model-implied excess returns “lag” the actual data.

## C.3 Alternative population specifications

Here we investigate the model’s robustness to the choice of  $v^A$ ; in particular, we show that the results are not affected much when  $v^A$  is constrained to take higher values, as is apparent in Table C.2.

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<sup>13</sup>Our results are similar when we consider five- or ten-year averages, or when we isolate business-cycle frequencies using band-pass filtering.

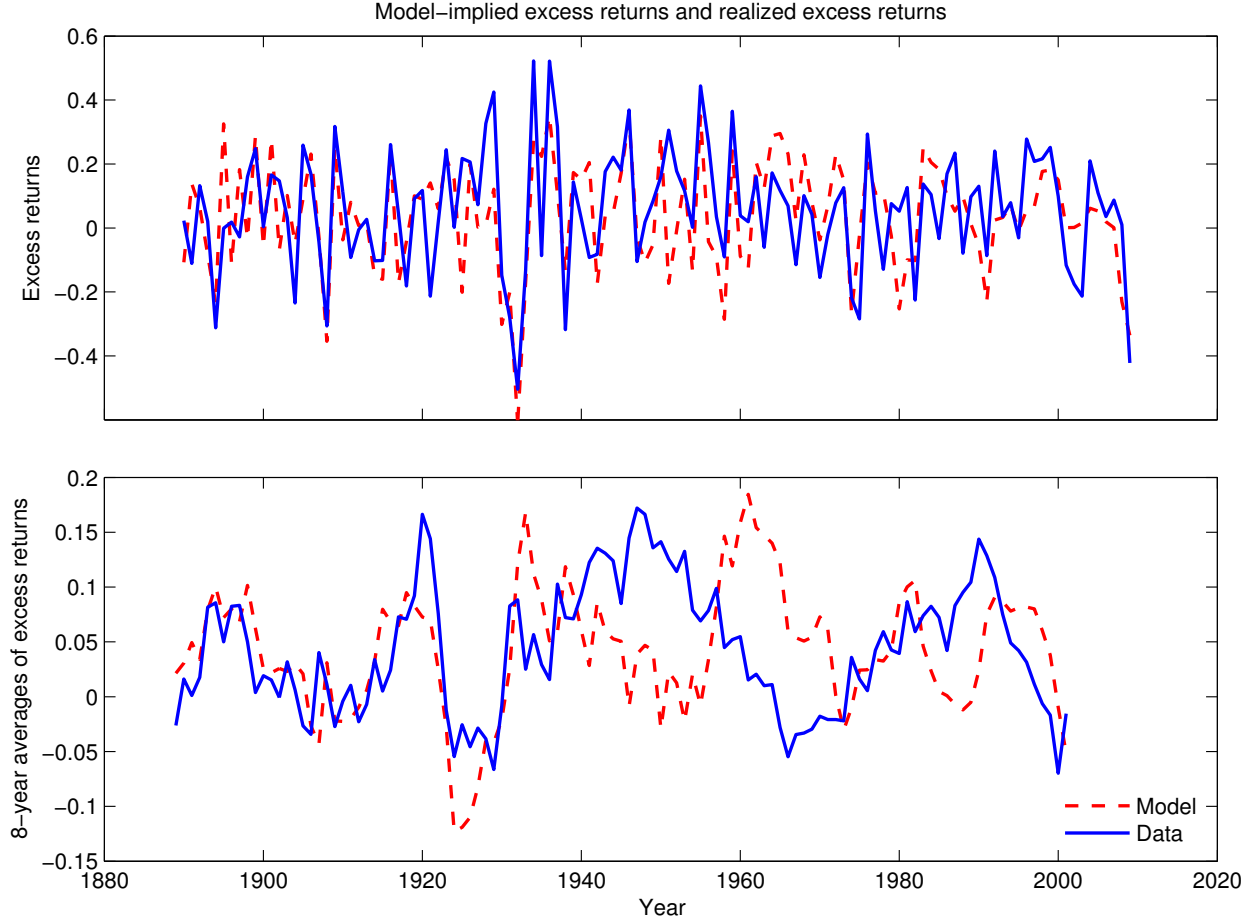


Figure C.1: Excess returns according to the model and realized excess returns in the data. To compute model-implied excess returns, we set the initial value of the consumption share of type- $A$  agents ( $X_t$ ) equal to its model-implied stationary mean. Since — according to the model — consumption growth is i.i.d., we determine consumption innovations by de-meaning first differences of log consumption growth from 1889 onwards. To account for the different standard deviations of consumption growth in the pre- and post- World War II sample we divide the de-meaned first differences of log consumption by the respective standard deviations of the two subsamples. The resulting series of consumption innovations, together with the initial condition  $X_0$ , is used to compute the model-implied path of  $X_t$ . Given  $X_t$ , the dotted line depicts the model-implied excess return for each year (top panel) and the corresponding annualized 8-year excess return (bottom panel). For comparison, the solid line depicts the data counterparts.

## D Recursive preferences with mortality risk

The purpose of this section is to show how the continuous-time objective (2) in the paper obtains as a limit when the time-interval becomes infinitesimal and agents' bequest motives disappear. The main issue has to do with the agent's attitudes towards mortality risk.

Parameter	Data	Baseline	CRRA	$v^A=0.08$
Risk aversion of type- <i>A</i> agents		1.5	1.5	1.5
Risk aversion of type- <i>B</i> agents		10	10	10
IES of type- <i>A</i> agents		0.7	$\frac{1}{\gamma^A}$	0.6
IES of type- <i>B</i> agents		0.05	$\frac{1}{\gamma^B}$	0.03
Population share of type- <i>A</i> agents		0.01	0.01	0.08
Equity premium	5.2 %	5.1 %	2.1 %	3.7 %
Volatility of returns	18.2%	17.0 %	8.5 %	16.1 %
Average (real) interest rate	2.8%	1.4%	2.1 %	2.3 %
Volatility of the real interest rate	0.92%	0.93%	0.13 %	0.86%

Table C.2: This table repeats the quantitative exercise of Table 1, except that we include an additional column where we restrict  $v^A = 0.08$  (in addition to  $\gamma^B \leq 10$ ) and determine the rest of the parameters to match (as closely as possible) the asset pricing moments at the bottom of the table. The choice  $v^A = 0.08$  is based on the fact that entrepreneurs account for 8% of the population. A potential motivation for matching the fraction of entrepreneurs is that entrepreneurs are likely to be more risk tolerant. In fact, an extended version of the model can generate this outcome endogenously. In a previous version of the paper, we included a time-zero occupational choice for an agent. We modeled an entrepreneur as someone who accepts a multiplicative shock (with mean 1) on her time-zero endowment of earnings in exchange for a non-pecuniary benefit of being self-employed. It followed that the relatively more risk-tolerant agents “self-select” into entrepreneurship. We also note that setting  $v^A = 0.1$  produces similar results to  $v^A = 0.08$ .

We start by postulating the following general objective function

$$\widehat{V}_t = \max_{c_t, \theta_t, W_{t+\Delta}^d} \left\{ (1 - e^{-\rho\Delta}) c_t^\alpha + e^{-\rho\Delta} \left( \mathbb{E}_t \left[ \left( \mathbb{E}_t \left[ 1_S \widehat{V}_{t+\Delta}^{1-\chi} + 1_D B(W_{t+\Delta}^d)^{1-\chi} \mid \mathcal{F}_{t^I} \right] \right)^{\frac{1-\gamma}{1-\chi}} \right] \right)^{\frac{\alpha}{1-\gamma}} \right\}^{\frac{1}{\alpha}}, \quad (\text{D.1})$$

where  $\Delta$  is the time interval,  $1_S$  equals 1 if the agent survives beyond  $t + \Delta$  and 0 otherwise,  $1_D = 1 - 1_S$ , and  $B(W_{t+\Delta}^d)^{1-\chi}$  is the utility the agent gets from bequeathing  $W_{t+\Delta}^d$  in case she dies at time  $t + \Delta$ . The filtration  $\mathcal{F}_{t^I}$  is explained below.

Objective (D.1) is a generalization of the objective considered in Epstein and Zin (1989) and Weil (1989), which obtains as a special case for  $\gamma = \chi$ . The difference is due to the two-layered expectation. The inner expectation conditions on the filtration  $\mathcal{F}_{t^I}$ , which is the information known to the agent at the intermediate stage of the period  $[t, t + \Delta]$ . We assume that at  $t^I$  the agent knows the realization of all time- $t+\Delta$  random variables, with the possible exception of whether she survives. If survival is assumed to be realized at the same time, then survival is also measurable with respect to this filtration; otherwise it is not.

The relevance of the first expectation layer and the definition of  $\mathcal{F}_{t^I}$  can be illustrated more transparently in the context of a simple example, involving the two lotteries depicted

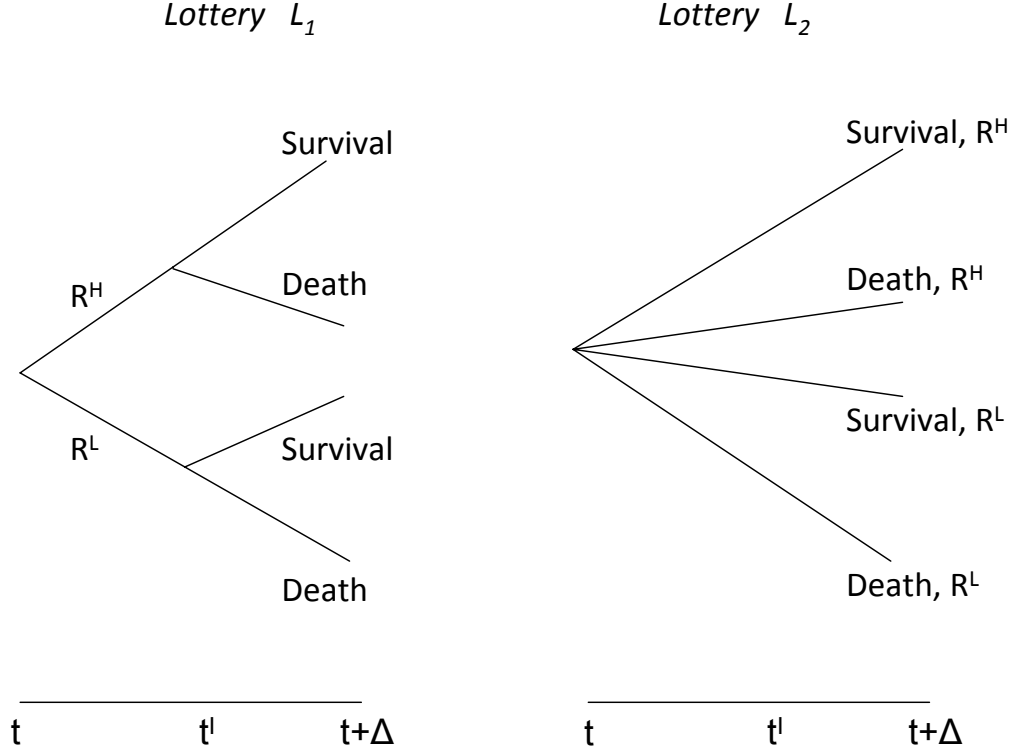


Figure D.1: Two lotteries illustrating the notion of learning one's survival status early ( $L_2$ ), respectively late ( $L_1$ ).

in Figure D.1. The example assumes that only two random variables are relevant: some binary “return”  $R$  (determining, for instance,  $W_{t+\Delta}$  and the investment set at time  $t + \Delta$ ), and whether the agent survives. In both cases,  $\mathcal{F}_{t'}$  contains the realization of  $R$ , but in the case of the sequential lottery  $L_1$  survival is not measurable with respect to  $\mathcal{F}_{t'}$ . In the case of the simultaneous lottery  $L_2$  it is, so that all variables inside the inner expectation operator are measurable with respect to the conditioning information.

It is natural to think of  $L_1$  as learning (about one's survival) late, and of  $L_2$  as learning early. Then Jensen's inequality implies that an agent has a preference for early resolution of the uncertainty *associated with death* ( $L_2$ ) if  $\frac{1-\gamma}{1-\chi} > 1$ , and a preference for late resolution of such uncertainty otherwise. Indeed, under  $L_1$ , respectively  $L_2$ , the argument of the outer expectation in objective (D.1) is written as

$$\left( e^{-\pi\Delta} \widehat{V}_{t+\Delta}^{1-\chi} + (1 - e^{-\pi\Delta}) B (W_{t+\Delta}^d)^{1-\chi} \right)^{\frac{1-\gamma}{1-\chi}}, \quad (\text{D.2})$$

respectively

$$e^{-\pi\Delta} \widehat{V}_{t+\Delta}^{1-\gamma} + (1 - e^{-\pi\Delta}) B (W_{t+\Delta}^d)^{1-\gamma}. \quad (\text{D.3})$$

Expression (D.3) is larger than (D.2) if  $x \mapsto x^{\frac{1-\gamma}{1-\chi}}$  is convex, i.e.,  $\frac{1-\gamma}{1-\chi} > 1$ .

The simple example illustrates that, when the independence axiom is not assumed, agents have preferences not only about the timing of the resolution of uncertainty, but also over the sequencing of the resolution of uncertainty about different issues. (See Gul and Ergin (2008), who use the term ‘‘preference for different issues’’).

A practical implication of the specification (D.1) is that, as long as survival is not in  $\mathcal{F}_{tI}$  (as in lottery  $L_1$ ), which we assume from now on, it becomes possible to disentangle aversion towards investment risks and death-related risks. For instance, if an agent faces no investment risk, then the utility function becomes

$$\widehat{V}_t = \max_{c_t, \theta_t, W_{t+\Delta}^d} \left\{ (1 - e^{-\rho\Delta}) c_t^\alpha + e^{-\rho\Delta} \left( e^{-\pi\Delta} \widehat{V}_{t+\Delta}^{1-\chi} + (1 - e^{-\pi\Delta}) B (W_{t+\Delta}^d)^{1-\chi} \right)^{\frac{\alpha}{1-\chi}} \right\}^{\frac{1}{\alpha}} \quad (\text{D.4})$$

Equation (D.4) shows that an agent’s incentive to smooth the marginal utility of a dollar across death and survival events is controlled by  $\chi$ . Similarly, for an infinitely lived agent ( $\pi = 0$ ) death risk disappears and objective (D.1) corresponds to the standard Epstein-Zin-Weil specification

$$\widehat{V}_t = \max_{c_t, \theta_t} \left\{ (1 - e^{-\rho\Delta}) c_t^\alpha + e^{-\rho\Delta} \left( E_t \left[ \widehat{V}_{t+\Delta}^{1-\gamma} \right] \right)^{\frac{\alpha}{1-\gamma}} \right\}^{\frac{1}{\alpha}}, \quad (\text{D.5})$$

which shows that  $\gamma$  controls the aversion towards investment risks.

Since in the text we are interested in the special case where agents have no bequest motives, we will henceforth set  $B (W_{t+\Delta}^d)^{1-\chi} = 0$ . In this case the Euler equation in the presence of competitively priced annuities and deterministic returns is given by

$$c_t^{\alpha-1} = e^{-\rho\Delta} (1 + r^f \Delta) (e^{-\pi\Delta})^{\frac{\alpha}{1-\chi}-1} c_{t+\Delta}^{\alpha-1}. \quad (\text{D.6})$$

To keep the model comparable to Blanchard (1985) and Yaari (1965), we make the assumption  $\alpha = 1 - \chi$ , so that Equation (D.6) becomes identical to the standard, textbook, Euler equation. With  $B (W_{t+\Delta}^d)^{1-\chi} = 0$  and  $\alpha = 1 - \chi$ , objective (D.1) simplifies to

$$\widehat{V}_t = \max_{c_t, \theta_t} \left\{ (1 - e^{-\rho\Delta}) c_t^\alpha + e^{-(\rho+\pi)\Delta} \left[ E_t \left( \widehat{V}_{t+\Delta}^{1-\gamma} \right) \right]^{\frac{\alpha}{1-\gamma}} \right\}^{\frac{1}{\alpha}}. \quad (\text{D.7})$$

From this point on, one can take the limit as  $\Delta \rightarrow 0$  and repeat the rigorous arguments in Duffie and Epstein (1992) to establish the correspondence between the discrete-time objective (D.7) and the continuous-time objective (2) in the main text. Here we follow a different route due to Obstfeld (1994). The argument in Obstfeld (1994) gives the essential intuition in a concise way, leaving out the technical issues that are addressed in Duffie and Epstein (1992).

To that end, we define

$$z(x) \equiv \frac{1-\gamma}{\alpha} x^{\frac{\alpha}{1-\gamma}}, \text{ and } V_t \equiv \rho^{\frac{\gamma-1}{\alpha}} \frac{\widehat{V}_t^{1-\gamma}}{1-\gamma}$$

We note that since  $V_t$  is a monotone transformation of  $\widehat{V}_t$ , the problems of maximizing  $\widehat{V}_t$  and  $V_t$  are equivalent, and (D.7) can be re-written as

$$z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right) = \max_{c_t, \theta_t} \left\{ \left( \frac{1-\gamma}{\alpha} \right) (1 - e^{-\rho\Delta}) c_t^\alpha + e^{-(\rho+\pi)\Delta} z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) E_t V_{t+\Delta} \right) \right\}. \quad (\text{D.8})$$

Equation (D.8) can now be written as

$$0 = \Delta \max_{c_t, \theta_t} \left\{ \begin{aligned} & \left( \frac{1-\gamma}{\alpha} \right) \left( \frac{1-e^{-\rho\Delta}}{\Delta} \right) c_t^\alpha + \frac{(e^{-(\rho+\pi)\Delta}-1) z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) E_t V_{t+\Delta} \right)}{\Delta} \\ & + \frac{z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) E_t V_{t+\Delta} \right) - z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right)}{\Delta} \end{aligned} \right\}.$$

Since  $\Delta$  is arbitrary, the maximized value of the expression inside curly brackets must be zero for any value of  $\Delta$ . Sending  $\Delta \rightarrow 0$ , applying the theorem of the maximum, and using  $\lim_{\Delta \rightarrow 0} E_t V_{t+\Delta} = V_t$  gives

$$0 = \max_{c, \pi} \left\{ \begin{aligned} & \left( \frac{1-\gamma}{\alpha} \right) \rho c_t^\alpha - (\rho + \pi) z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right) \\ & + z' \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right) \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) \lim_{\Delta \rightarrow 0} \frac{E_t (V_{t+\Delta} - V_t)}{\Delta} \end{aligned} \right\}. \quad (\text{D.9})$$

Now divide though by  $z' \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right) \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma)$  and note that

$$\frac{z \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right)}{\rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) z' \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right)} = \frac{1-\gamma}{\alpha} V_t, \quad (\text{D.10})$$

and

$$\frac{\left( \frac{1-\gamma}{\alpha} \right) \rho c_t^\alpha}{z' \left( \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma) V_t \right) \rho^{\frac{1-\gamma}{\alpha}} (1-\gamma)} = \frac{1}{\alpha} \frac{c_t^\alpha}{\left( (1-\gamma) V_t \right)^{\frac{\alpha}{1-\gamma}-1}}. \quad (\text{D.11})$$

Combining (D.9) with (D.10) and (D.11) gives the Bellman equation

$$0 = \max_{c, \pi} \left\{ \frac{1}{\alpha} \left( \frac{c_t^\alpha}{\left( (1-\gamma) V_t \right)^{\frac{\alpha}{1-\gamma}-1}} - (\rho + \pi) (1-\gamma) V_t \right) + \lim_{\Delta \rightarrow 0} \frac{E_t (V_{t+\Delta} - V_t)}{\Delta} \right\},$$

which corresponds to the Bellman equation implied by equations (1) and (2) in the text.

## References

- Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance* 59(4), 1481–1509.
- Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy* 93(2), 223–247.
- Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy* 107(2), 205–251.
- Campbell, J. Y. and N. G. Mankiw (1989). *Consumption, Income, and Interest Rates: Reinterpreting the Time Series Evidence.*, pp. 185–216. Princeton U and NBER: Cambridge, Mass. and London:.
- Cutler, D. M. and L. F. Katz (1992). Rising inequality? changes in the distribution of income and consumption in the 1980's. *American Economic Review* 82(2), 546–551.
- Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica* 60(2), 353–394.
- Epstein, L. G. and S. E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica* 57(4), 937–69.
- Gul, F. and H. Ergin (2008). A subjective theory of compound lotteries. Working Paper, Princeton and Washington University in St. Louis.
- Hall, R. E. (1988). Intertemporal substitution in consumption. *Journal of Political Economy* 96(2), 339–357.
- Hubbard, R. G., J. Skinner, and S. P. Zeldes (1994). The importance of precautionary motives in explaining individual and aggregate saving. *Carnegie-Rochester Conference Series on Public Policy* 40, 59–125.
- Malloy, C., T. Moskowitz, and A. Vissing-Jorgensen (2009). Long-run stockholder consumption risk and asset returns. *Journal of Finance* 64(6), 2427–2479.
- Obstfeld, M. (1994). Risk-taking, global diversification, and growth. *American Economic Review* 84(5), 1310–1329.
- Schroder, M. and C. Skiadas (1999). Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory* 89(1), 68–126.
- Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics* 24(3), 401–421.
- Wolff, E. N. (2010). Recent trends in household wealth in the united states: Rising debt and the middle-class squeeze - an update to 2007. Working Paper No. 589. Annandale-on-Hudson, NY: The Levy Economics Institute of Bard College.
- Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies* 32, 137–50.