Auctions with Endogenous Selling*

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Abstract

The seminal paper by Milgrom and Weber (1982) ranks the expected revenues of several auction mechanisms, taking the decision to sell as exogenous. We endogenize the sale decision. The owner decides whether or not to sell, trading off the conditional expected revenue against his own use value, and buyers take into account the information contained in the owner's sale decision. We show that revenue ranking implies volume and welfare ranking under certain general conditions. We use this to show that, with affiliated signals, English auctions have larger expected price, volume, and welfare than second-price auctions, which in turn have larger expected price, volume, and welfare than first-price auctions.

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1 Introduction

The classic auction theory[1] studies the expected revenue to an owner committed to selling — that is, the sale decision is taken as exogenous. This is a natural viewpoint, for instance, for a government selling an oil field, where the sale decision is unrelated to the quality of the field.

The decision to sell by the owner of a private company, on the other hand, depends on a trade-off between the owner's benefits from running the company and the revenue raised by the sale. Further, potential buyers take into account the (adverse) information revealed by the decision to sell. These considerations are also relevant for the sale of a block of shares in a public company, the sale of a house, the resale of an oil field, and many others.

The endogenous sale decision raises several new questions: How does the auction mechanism affect the owner's expected revenue conditional on his private signal (not just the unconditional expected revenue as in the standard models)? How does the auction mechanism affect the equilibrium set of signals that lead to a sale, hence the volume of trade? How does the trading mechanism affect the average observed revenue, taking into account the endogeneity in the sale decision? Finally, what is the effect on welfare as captured by the allocational efficiency of the sale decision?

This paper addresses these questions by introducing endogenous sales in the seminal model of Milgrom and Weber (1982). In the model, an owner and n potential buyers receive private signals. These $n + 1$ signals are affiliated, which, informally, means that if one agent's signal is good then the other agents' signals are likely to be good, as well. The asset's value to an agent depends positively on his own signal as well as the other signals, and it has both common- and private-value components.

Milgrom and Weber (1982) show that with affiliated signals (and an exogenous sale) the expected revenue of an English auction is higher than that of a second-price

[1] Important papers include Vickrey (1961), Myerson (1981), Riley and Samuelson (1981), and Milgrom and Weber (1982). For an overview of auction theory see Klemperer (2000), Krishna (2002), and Milgrom (2004).

auction, which, in turn, has a higher expected revenue than a first-price auction. We give general conditions under which this "revenue ranking" between two auction mechanisms for a fixed anticipated sale set implies "volume ranking," "endogenous-sale revenue ranking," and "welfare ranking." By this, we mean that (1) the owner sells on a larger set of signals when using the auction mechanism that generates higher expected revenue *for a fixed (anticipated) sale set*, (2) this larger sale set reinforces the revenue ranking, and (3) it leads to a more efficient allocation of the asset. We use these general results to show specifically that an English auction has larger expected prices, volume, and welfare than a second-price auction, which in turn has larger expected prices, volume, and welfare than a first-price auction, under certain conditions.

To see the intuition behind these results, consider, for instance, a first-price auction with an equilibrium set X of owner signals that lead to sales. If the owner used a second-price auction instead, and the bidders still expected X to be the sale set, then the expected price conditional on any owner signal would be higher. Hence, the owner would still want to sell for signals in X, and also if he had (slightly) better signals, so he would sell on a larger set $X^2 \supset X$ of signals. If the bidders anticipated the sale set X^2, then they would bid more. Taking this into account, the owner would sell on an even larger set $X^3 \supset X^2$ of signals, and so on. The limit $\cup_i X^i$ is the equilibrium sale set for the second-price auction. This sale set is clearly larger than the first-price-auction sale set, and the second price auction has a larger expected revenue conditional on an observed sale.

The argument above uses the fact that, for a given (anticipated) sale set, the second-price auction has a higher expected price than the first-price auction *conditional on any signal the owner might have.* This is a stronger result than that of Milgrom and Weber (1982), who show that the price is higher *when averaging over the owner's signals.* We show that this "strong revenue ranking" applies when comparing the first- and second-price auctions. The strong revenue ranking does not apply, however, when comparing the English and second-price auctions. The English auction has a smaller winner's curse, which increases expected prices (Milgrom and Weber (1982)), but it also has the effect of partially revealing the owner's signal in the course of the auction. Therefore, conditional on a low owner signal the expected price in an English auction can be lower than that of a second-price auction.[2] For high owner signals, on the other hand, we have the standard ranking of expected prices. The highest owner signal in the sale set is the signal for which the owner is indifferent between selling and not selling. This "marginal" signal determines the equilibrium, and since revenue ranking applies conditional on this signal, the English auction has a larger equilibrium sale set than the second-price auction.

In addition to this volume ranking, we also rank mechanisms in terms of welfare. The intuition for welfare being higher with a higher-volume mechanism is as follows: When the owner decides to sell, the expected price is higher than his expected utility of keeping the asset. Further, the buyer expects a higher utility than the price. Therefore, the buyer's utility of owning is higher than the seller's, and the trade is welfare improving. This intuition over-simplifies the problem slightly by ignoring conditioning information, but, under certain conditions, it is correct at least for high owner signals, implying that a higher-volume mechanism indeed is associated with higher welfare.

In the simple benchmark in which agents have independent private signals, the owner's decision to sell and expected prices are the same for a large class of auction mechanisms. This is a straightforward extension of the Revenue Equivalence Theorem (Vickrey (1961), Myerson (1981), Riley and Samuelson (1981)) to the case of an endogenous sale decision.

Lastly, our results generalize in a simple way to a dynamic setting with short-lived private information, which clarifies our interpretation of sale sets as related to the volume of trade. There exist several other important dynamic extensions of the standard auction model, which focus mainly on the interaction between the information

[2] A parametric example that illustrates this lack of strong revenue ranking between English and second-price auctions is available upon request from the authors.

revealed through bidding in a "primary market" and the price in a "secondary market" (Bikhchandani and Huang (1989), Ausubel and Cramton (1999), Haile (1999, 2001, 2003), Zheng (2002), Nyborg and Strebulaev (2004)). We complement these papers by studying the effects associated with the owner's private information.

To summarize, this paper studies an endogenous-sale framework with many natural applications and shows that some trading mechanisms create more "liquid" markets with higher volume, prices, and welfare than other mechanisms.

2 Model

We consider a one-period game played by a set $\mathcal{N} = \{0, 1, \ldots, n\}$ of risk-neutral agents. There is a single object, initially owned by agent 0. Before any action is taken, each agent i receives a private signal x^i. The signals $(x^i)_{i\in\mathcal{N}}$ are symmetrically distributed[3] with a positive density on $\mathbb{X} = \{x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}\}$. The signals predict the value of the security to the owner at the end of the game. Specifically, the value to agent i is the random variable $V^i \in \mathbb{R}$, defined on the probability space $(\Omega, \mathcal{F}, Pr)$, with

$$E(V^i \mid x^0, \ldots, x^n) = v(x^i, x^0, (x^j)_{j\neq 0,i}), \tag{1}$$

for $i > 0$ and

$$E(V^0 \mid x^0, \ldots, x^n) = u(x^0, (x^j)_{j>0}), \tag{2}$$

where $v, u : \mathbb{X}^{n+1} \to \mathbb{R}$ are increasing in all arguments, v is symmetric in its last $n-1$ arguments[4], and u is symmetric in its last n arguments.

After receiving his signal, the owner decides whether to keep the object or offer it

[3] This means that for any permutation, ρ, of $\{0, \ldots, n\}$, $(x^0, \ldots, x^n)$ and $(x^{\rho(0)}, \ldots, x^{\rho(n)})$ are identically distributed.

[4] That is, for any permutation, ρ, of $\{1, \ldots, n-1\}$, $v(a, b, c_1, \ldots, c_{n-1}) = v(a, b, c_{\rho(1)}, \ldots, c_{\rho(n-1)})$, and similarly for u.

for sale. If the owner decides to sell, the object is sold using an auction mechanism M.

An ***auction mechanism*** $M = (\Theta, \pi, z)$ consists of a measurable space Θ is the set of allowed "bids" and two symmetric mappings π, z. For sealed-bid auctions, a bid is just a real number, but for other auction mechanisms a "bid" can be a complicated strategy. In an English Auction, for example, a "bid" is a specification of the price level at which to drop out conditionally on the price levels at which other bidders have dropped out. To an n-tuple of bids, $\pi : \Theta^n \to \{(x_1, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$ assigns the probabilities with which each of the bidders acquires the object, and $z : \Theta^n \to \mathbb{R}^n$ assigns the amounts to be paid by the bidders. We assume that it is possible to bid "zero" and be sure to pay zero, i.e., there exists $\underline{\theta}$ such that $z^1(\underline{\theta}, \theta_2, \ldots, \theta_n) = 0$ for all $\theta_2, \ldots, \theta_n$. The price received by the seller, $p : \Theta^n \to \mathbb{R}$, is the sum of the transfers from the bidders: $p = \sum_{i=1}^n z_i$.[5]

A strategy for the seller is a set $X \subset \mathbb{X}$ of signals for which he sells the object. For the other agents, it is a bidding strategy $b : \mathbb{X} \to \Theta$. The utility, Π^i, of agent i is given by

$$\Pi^i(X, b^1 \ldots, b^n) = E\left(V^i 1_{(i=o)} - \bar{z}^i\right),$$

where $o \in \mathcal{N}$ is the owner of the object at the end of the game and $\bar{z}^i$ is the net cash payment (receipt if negative) made by agent i. That is, if there is a sale, $\bar{z}^i = z^i$ for $i \neq 0$ and $\bar{z}^i = -p$ for $i = 0$; otherwise, $\bar{z}^i = 0$ for all i.

Definition 1 *An* **equilibrium** *is a sale set $X \subset \mathbb{X}$ and bidding strategies $b^1, \ldots, b^n$,*

[5]We do not allow for reserve prices, which would also reflect the owner's information. This would raise questions of existence and monotonicity outside the scope of this paper. Furthermore, it may be natural to abstract from reserve prices if the owner is unable to commit to them. Also, the owner has no incentive to set a reserve price if he were to sell to an uninformed intermediary who would then run an auction for the other agents.

such that

$$\Pi^0(X, b^1 \ldots, b^n) \geq \Pi^0(X', b^1 \ldots, b^n)$$
$$\Pi^i(X, b^1 \ldots, b^n) \geq \Pi^i(X, b^1 \ldots, b^{i-1}, b', b^{i+1}, \ldots b^n)$$

for all $i > 0$*,* $X' \subset \mathbb{X}$*, and bidding strategies* b'*.*

A **symmetric equilibrium** *is an equilibrium in which all non-owners use the same bidding strategy* $b : \mathbb{X} \to \Theta$*.*

Our goal is to incorporate sale endogeneity in the analysis of Milgrom and Weber (1982), rather than to add to the study of bidding in standard auctions. We consequently focus on mechanisms that admit a symmetric equilibrium for each sale set X and where the winner is the bidder with the highest signal x^i, denoted ***natural mechanisms***. Many standard auction mechanisms are natural mechanisms under standard distributional assumptions — see, for instance, Milgrom and Weber (1982), Klemperer (2000), and references therein.

3 Ranking of Price, Volume, and Welfare

Our main contribution is to show how an auction mechanism affects equilibrium allocations, welfare, and prices, building on the work of Milgrom and Weber (1982). We first provide a set of general conditions that capture the intuition behind the results, and then show that these conditions are met for the first-price, second-price, and English auctions under natural distributional assumptions.

3.1 General Results

It is useful to introduce some notation for a natural mechanism M and an associated fixed equilibrium. First, let the price paid to the seller be $p^M(x^1, \ldots, x^n, X)$, where

X is the sale set and bidders have signals $x^1, \ldots, x^n$. Second,

$$P^M(x^0, X) = E\left[p^M(x^1, \ldots, x^n, X) \mid x^0\right]$$

is (a.s.) the conditional expected price, given the owner's signal, x^0. Third,

$$\bar{u}(x^0) = E\left[u(x^0, \ldots, x^n) \mid x^0\right]$$

is the expected value associated with keeping the asset for an owner with signal x^0, and, fourth,

$$\bar{v}(x^0) = E\left[v(x^{(1)}, x^0, (x^{(j)})_{j\geq 2}) \mid x^0\right]$$

is the owner's belief about the asset value to the bidder with highest signal (the winner of the auction), and $x^{(k)}$ denotes the k^{th} highest among $x^1, \ldots, x^n$.

We are now ready to list the set of general conditions on the equilibria that guarantee the ranking results. We verify in the next section that these natural conditions hold for first price, second price, and English auctions when signals are affiliated.

Condition 1 *For all $X \in \mathbb{X}$, the set $\{x^0 \in \mathbb{X} : \bar{u}(x^0) \leq P^M(x^0, X)\}$ is either of the form $[\underline{x}, a]$ for some $a \in \mathbb{X}$, or empty.*

An owner with private signal x^0 wants to sell if and only if the expected price $P^M(x^0, X)$ is greater than the value of keeping the asset, $\bar{u}(x^0)$. Hence, the first condition states that the owner chooses to sell on a connected subset of $\mathbb{X}$ containing its smallest element. A second natural condition is that enlarging the sale set by adding higher signals increases the expected price:

Condition 2 *For any, $x^0 \in \mathbb{X}$, and (a, b) such that $a < b \leq \overline{x}$, it holds that $P^M(x^0, [\underline{x}, a]) \leq P^M(x^0, [\underline{x}, b])$.*

Finally, in order to make welfare comparisons, the third condition requires that, conditional on the owner having the best possible selling signal, the expected price is lower than the winning bidder's utility.

Condition 3 *For all* $a \in \mathbb{X}$, $P^M(a, [\underline{x}, a]) \leq \bar{v}(a)$.

This condition is natural in part because the bidders extract rents and in part because the bidders face an adverse-selection problem and expect the owner's signal to be lower than a.

We are ready to make precise our claim that revenue ranking implies volume and welfare ranking:

Theorem 1 *Consider two natural mechanisms,* M *and* N*, satisfying Conditions 1–2, such that*

$$P^M(a, [\underline{x}, a]) \geq P^N(a, [\underline{x}, a]) \tag{3}$$

for all $a \in \mathbb{X}$*. If* $\{(X^M, b^M)\}$ *is a symmetric equilibrium for* M*, then there exists a symmetric equilibrium,* $\{(X^N, b^N)\}$*, for* N *with* $X^N \subseteq X^M$*. Conversely, if* $\{(X^N, b^N)\}$ *is a symmetric equilibrium for* N*, then there exists a symmetric equilibrium,* $\{(X^M, b^M)\}$*, for* M *with* $X^M \supseteq X^N$*. The equilibrium* $\{(X^M, b^M)\}$ *has a higher revenue than* $\{(X^N, b^N)\}$*, that is,*

$$E\left(P^M(x^0, X^M) \mid x^0 \in X^M\right) \geq E\left(P^N(x^0, X^N) \mid x^0 \in X^N\right) \tag{4}$$

If, in addition, $\bar{u}(x^0) - \bar{v}(x^0)$ *increases in* x^0 *and* M *satisfies Condition 3, welfare in the* M*-equilibrium is higher than that in the* N*-equilibrium.*

From now on, we shall find it convenient to say, whenever the conclusion of the theorem regarding the existence and comparison of the sale sets of equilibria holds, that mechanism M has ***higher volume*** than N. If (4) holds we say that M has ***higher endogenous-sale expected price*** than N, and, if the conclusion regarding welfare holds, we say that M has ***higher welfare*** than N.

The premise of the theorem, inequality (3), states that for any fixed anticipated sale set, mechanism M yields a higher expected revenue than mechanism N, conditionally on the owner's signal being the best possible signal for which he sells. The intuition for the volume ranking is as follows. Suppose that $[\underline{x}, a^1]$ is an equilibrium sale set for mechanism N. Consider the owner's best response if the trading mechanism is changed to M, but bidders keep anticipating the same sale set. Then, because of the revenue ranking, the owner will sell for a larger set of signals, say $[\underline{x}, a^2]$. Now, suppose that the bidders anticipate the sale set $[\underline{x}, a^2]$. Then, by Condition 2, they will bid more, since now they think that the owner might have a better signal. This, in turn, will lead the owner to sell with even better signals, and by iterating this argument, (in the limit) an equilibrium obtains that has larger sale set and expected price than for mechanism N.

The intuition for the welfare result is also very simple. Condition 3 implies that trade is welfare improving given that the owner has the highest signal for which he would sell. This is because the owner's utility is smaller than the price, which in turn is smaller than the buyer's utility. If, in addition, $\bar{u}(x^0, X) - \bar{v}(x^0, X)$ increases in x^0, then trade is welfare improving for all equilibrium sales, whence a smaller sale set is welfare reducing.

We note that is it natural to assume that $\bar{u}(x^0) - \bar{v}(x^0)$ is increasing in x^0, since this means that the owner's expected utility depends more strongly on his signal than does the winning bidder's expected utility. The assumption is satisfied, for instance, if the signals are distributed according to a (truncated) normal distribution with all correlations positive — implying affiliation — and if

$$\begin{aligned} u(z_0, \ldots, z_n) &= \alpha z_0 + h(z_0, \ldots, z_n) \\ v(z_0, \ldots, z_n) &= \beta z_0 + h(z_0, \ldots, z_n), \end{aligned} \tag{5}$$

where h is symmetric and increasing in the first $n+1$ arguments and $\alpha \geq \beta > 0$.

Finally, a simple benchmark is provided by the case in which private signals are independent. The standard Revenue Equivalence Theorem generalizes to our endogenous-trade setting in a straightforward way. Intuitively, the following corollary states that all natural mechanisms have the same equilibrium sale decisions by owners — hence, the same trading volume and welfare — and the same average prices.

Corollary 1 *Assume that the private signals x^i are independent across agents and consider two natural mechanisms M and N. If $\{(X, b^M)\}$ is an equilibrium for M then there exists b^N such that $\{(X, b^N)\}$ is a symmetric equilibrium for N. These equilibria have the same sale sets and expected prices.*

3.2 First-Price, Second-Price, and English Auctions

Milgrom and Weber (1982) show that the average revenues yielded by first-price and second-price auctions can be ranked when bidders' signals are affiliated.[6] To examine how this result generalizes to our setting, we assume throughout the remainder of the paper that the random variables $x^0, x^1, \ldots, x^n$ are affiliated. In order for the results of Milgrom and Weber (1982) to be applicable, though, bidders' signals need to be affiliated *conditional on the event that the owner sells,* which is implied by our next result.

Lemma 1 *Suppose $Z_0, Z_1, \ldots, Z_n$ are affiliated random variables in $\mathbb{R}$, and A is a measurable subset of $\mathbb{R}$ with $Pr(Z_0 \in A) > 0$. Then, given $Z_0 \in A$, $Z_1, \ldots, Z_n$ are affiliated.*

We aim to use Theorem 1 to rank the three auction mechanism in terms of revenue, volume, and welfare. To that end, we verify that Conditions 1-3 are satisfied by these mechanisms.

Lemma 2 *Each of the first-price, second-price, and English auctions has an equilibrium that satisfies Conditions 2 and 3. These equilibria also satisfy Condition 1 if*

$$E\left(\frac{\partial u}{\partial x^0} \;\middle|\; x^0\right) > k\; E\left(\left(\frac{\partial \log \zeta}{\partial x^0}\right)^2 \;\middle|\; x^0\right)^{1/2} \tag{6}$$

[6] See Milgrom and Weber (1982) for a definition of affiliation.

where ζ *is the conditional density of* $(x^1, \ldots, x^n)$ *given* x^0*, and*

$$k = E\left(\left(v(x^{(1)}, \overline{x}, x^{(1)} \ldots, x^{(1)})\right)^2 \,\middle|\, x^0\right)^{1/2}.$$

While Conditions 2-3 hold generally, Condition 1 does not. It is not difficult, however, to provide sufficient conditions under which it holds. Lemma 2 formalizes the intuition that, if the statistical dependence of signals is low enough relative to the slope of u with respect to x^0, then the owner's expected utility $\bar{u}$ depends more strongly on his signal x^0 than the expected price P^M. This implies that $\bar{u}(x^0) - P^M(x^0) = 0$ has at most one solution, and therefore Condition 1 holds. We note that both (6) and Condition 1 are clearly true if the owner's signal x^0 is independent of $(x^1 \ldots, x^n)$.

In order to apply Theorem 1 above, we must show that (3) holds, that is, that revenue ranking holds *conditional on an owner's highest signal that would make him sell.* (We note that this is different from the revenue ranking *on average* across an owner's signals as in Milgrom and Weber (1982).) When comparing the first-price and second-price auctions, we get an even stronger result, namely that revenue ranking holds conditionally on *any* seller signal.

Theorem 2 *For all* $x^0 \in \mathbb{X}$ *and* $X \subset \mathbb{X}$*,*

$$P^{II}(x^0, X) \geq P^I(x^0, X), \qquad (7)$$

where P^I *and* P^{II} *are the expected prices in the symmetric equilibria of the first-price and second-price auctions, respectively.*

Hence, the following corollary follows immediately from our previous results.

Corollary 2 *If (6) holds then the second-price auction has higher endogenous-sale expected price and volume than the first-price auction. If, in addition,* $\bar{u}(x^0) - \bar{v}(x^0)$ *increases in* x^0*, then the second-price auction has higher welfare than the first-price auction.*

Now, we turn to the comparison of the second-price and English auctions. Milgrom and Weber (1982) show that the winner's curse is smaller for the English auction, and therefore that the revenue is higher on average across the seller's signals. While this revenue ranking may actually *not* apply conditional on all seller signals, it does apply for the owner's highest selling signal, as required in Theorem 1, (3). The intuition is as follows. During the English auction, the information of those bidders with low signals is revealed. This not only diminishes the winner's curse, but also reveals information about the owner's signal. If the owner has a low signal, then this information revelation is bad for him. If he has the best possible selling signal, however, the information revelation is good for the seller, and reinforces the (conditional) revenue ranking.

Theorem 3 *For all $a \in \mathbb{X}$,*

$$P^{E}(a, [\underline{x}, a]) \geq P^{II}(a, [\underline{x}, a]) \,, \tag{8}$$

where P^{II} and P^{E} are the expected prices in the symmetric equilibria of the second-price and English auctions, respectively.

These results yield the following corollary.

Corollary 3 *If (6) holds then the English auction has higher endogenous-sale expected price and volume than the second-price auction. If, in addition, $\bar{u}(x^0) - \bar{v}(x^0)$ increases in x^0, then the English auction has higher welfare than the second-price auction.*

We briefly note how the revenue, volume, and welfare ranking of the three auction mechanisms studied here extends to a setting in which the owner sells to an uninformed intermediary, who then resells to the informed bidders. In that case, Condition 1 is satisfied trivially, since the price does not depend on the owner's information, and the theorem is true as stated. Furthermore, the sale set is always smaller than the one that obtains when the owner sells directly, which implies a lower welfare.

4 Repeated Auctions

Our model and results generalize easily to a multi-period setting in which, each period, the stage game is the same as in the main body of the paper: public and private signals are received, the current owner decides whether to sell or not, and then a potential sale takes place using an auction mechanism. The only requirement is that, conditional on the public signal, the distribution of private signals is independent of earlier signals — in other words, information is short lived.

In such a setup, bidders bid as in a single-period auction in which the "prize" is the next dividend plus the continuation-value of being an owner net of the value of being a buyer (similarly to e.g. Haile (2003)). Consequently, with affiliated private signals conditional on the public signal, the English auction has higher volume and welfare than the the second-price auction in all periods, while the second-price auction dominates the first-price auction in the same sense. With conditionally independent private signals, any two natural mechanisms have equilibria that are equivalent in terms of expected prices, allocations, and welfare in all periods.

A Appendix

Proof of Theorem 1: Let the sale set associated with mechanism N be $X^N = [\underline{x}, a^N]$, with $a^N \in \mathbb{X}$. We define $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ by $\mathcal{S}(x) = z$, where z satisfies $P^M(z, [\underline{x}, x]) = \bar{u}(z)$ if $P^M(\overline{x}, [\underline{x}, x]) \geq \bar{u}(\overline{x})$, and $z = \overline{x}$ otherwise. That is, if buyers anticipate the sale set $[\underline{x}, x]$ then the owner will sell on the set $[\underline{x}, \mathcal{S}(x)]$. Hence, a sale set $[\underline{x}, x]$ is and equilibrium if x is a fixed point for $\mathcal{S}$, that is, if $\mathcal{S}(x) = x$. We note that $\mathcal{S}$ is well-defined because of Condition 1. We are looking for a fixed point, a^M, which is larger than a^N. It follows from Condition 2 that $\mathcal{S}$ is (weakly) increasing. Further, it follows from (3) that $\mathcal{S}(a^N) \geq a^N$, and it is obvious that $\mathcal{S}(\overline{x}) \leq \overline{x}$. Now, Lemma 3 (below) implies that $\mathcal{S}$ has a fixed point in $[a^N, \overline{x}]$. One proves analogously that, given an equilibrium for M with sale set $[\underline{x}, a^M]$, an equilibrium exists for N with a smaller sale set $[\underline{x}, a^N]$.

The welfare claim follows immediately. The welfare is higher for mechanism M than for mechanism N if and only if

$$E[(\bar{v}(x^0) - \bar{u}(x^0))1_{x^0 \in (a^N, a^M]}] \geq 0.$$

The integrand is positive at $x^0 = a^M$ by Condition 3, and decreasing by assumption. Consequently, the expectation is positive.

Lemma 3 *Suppose that $f : [a, b] \to \mathbb{R}$, for $a, b \in \mathbb{R}$, is (weakly) increasing, and that $f(a) \geq a$ and $f(b) \leq b$. Then, f has a fixed point in $[a, b]$.*

Proof: This lemma is a special case of the lattice fixed-point theorem of Knaster-Tarski. See, for instance, Dugundji and Granas (1982). We provide a simple proof for the readers' convenience.

We may assume that $f(b) < b$. Then, $z = \inf\{x \in [a, b] : f(x) < x\}$ is well-defined. We claim that z is a fixed point. Consider a sequence, (y_i), in $\{x \in [a, b] : f(x) < x\}$ converging to z and, an increasing sequence, (z_i), also converging to z.

Then, we have the inequalities

$$z_i \leq f(z_i) \leq f(z) \leq f(y_i) < y_i,$$

and the proof is completed by letting i approach infinity.

□

Proof of Corollary 1:
It is immediate that Conditions 1-2 are satisfied when signals (x^j) are independent. The Revenue Equivalence Theorem ensures that (3) and its converse hold, which implies the result.

□

Proof of Lemma 1:
Let $Z = (Z_1, \ldots, Z_n)$. Consider any nondecreasing function $g : \mathbb{R}^n \to \mathbb{R}$, and any sublattice S of $\mathbb{R}^n$. Then, since a product of sublattices is a sublattice, and since $Z_0, Z_1, \ldots, Z_n$ are affiliated, Theorem 23, $(i) \Rightarrow (ii)$, of Milgrom and Weber (1982) gives

$$\begin{aligned} E\left[g(Z)h(Z) \;\middle|\; Z \in S, Z_0 \in A\right] \geq \\ E\left[g(Z) \;\middle|\; Z \in S, Z_0 \in A\right] E\left[h(Z) \;\middle|\; Z \in S, Z_0 \in A\right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} E^{(Z_0 \in A)}\left[g(Z)h(Z) \;\middle|\; Z \in S\right] \geq \\ E^{(Z_0 \in A)}\left[g(Z) \;\middle|\; Z \in S\right] E^{(Z_0 \in A)}\left[h(Z) \;\middle|\; Z \in S\right], \end{aligned}$$

where $E^{(Z_0 \in A)}$ denotes expectation with respect to the conditional distribution given $(Z_0 \in A)$. The latter inequality shows, using Theorem 23, $(ii) \Rightarrow (i)$, of Milgrom and Weber (1982), that $Z_1, \ldots, Z_n$, conditional on $(Z_0 \in A)$, are affilated.

□

Notation:

In the following proofs, we make use of some results in Milgrom and Weber (1982), to which we refer as "MW." For this reason we use notation that is very close to that of MW. The analysis relies on Lemma 1, which implies that the random variables $x^1, \ldots, x^n$ are affiliated conditionally on a sale, that is, $x^0 \in X$. We isolate bidder 1 and let $Y_1, \ldots, Y_{n-1}$ be the bids of the other $n-1$ bidders, arranged in *descending* order.

Proof of Lemma 2:

It is easy to see that Condition 3 is satisfied for the English auction, and hence for the first-price and second-price auctions by virtue of Theorems 2 and 3. Condition 2 requires a more involved argument.

First, we show that Condition 2 applies for the second-price auction. This follows from the fact that the equilibrium bids, which are given by

$$E\left(v(x, x^0, x, (Y_j)_{j>2}) \;\middle|\; x^1 = x, Y_1 = x, x^0 \le a\right),$$

increase in a, which is an immediate consequence of Theorem 5 in Milgrom and Weber (1982).

Second, showing that Condition 2 applies for the English auction is analogous to the argument given for the second-price auction.

Lastly, we show that Condition 2 holds for the first-price auction. We work under the (unrestrictive) assumption that the optimal bids in the first-price auction, denoted by $b(x, X)$, where x is the bidder's signal and X the sale set, are differentiable in the first argument. Let b_1 designate the derivative of b with respect to the first argument. MW show that the optimal bid in the first-price auction must obey the differential equation

$$b_1(x,X) = (\hat{v}(x,x,X) - b(x,X))\,\frac{f_{Y_1}(x|x,X)}{F_{Y_1}(x|x,X)}, \tag{A.1}$$

where $f_{Y_1}(\cdot|z,X)$, respectively $F_{Y_1}(\cdot|z,X)$ is the probability density function, respectively cumulative distribution function, of Y_1 conditionally on $x^1 = z$ and on the owner's signal being in the sale set X, and

$$\hat{v}(x,X) = E\left(v(x^1, x^0, (x^j)_{j\notin\{0,1\}}) \;\middle|\; x^1 = x, Y_1 = y, x^0 \in X\right).$$

Now, let a and a' be such that $\underline{x} \le a' < a \le \overline{x}$. Then, the equilibrium condition $b(\underline{x}, X) = \hat{v}(\underline{x}, \underline{x}, X)$ and Theorem 5 in MW imply that $b(\underline{x}, [\underline{x}, a']) \le b(\underline{x}, [\underline{x}, a])$. Suppose, in order to apply Lemma 5 below that $b(x, [\underline{x}, a]) \le b(x, [\underline{x}, a'])$ for some x. Then, by (A.1) and Lemma 4 (below), we conclude that $b_1(x, [\underline{x}, a']) \le b_1(x, [\underline{x}, a])$. Hence, using Lemma 5 we see that Condition 2 is also satisfied by the first-price auction.

Finally, in order to show that (6) is sufficient for Condition 1 to hold, we prove that $\bar{u}(x^0) - P^M(x^0)$ is a strictly increasing function of x^0. Evaluating the expressions below at $\hat{x}^0 = x^0$, we have

$$\begin{aligned}
\frac{\partial \bar{u}(\hat{x}^0)}{\partial \hat{x}^0} &= E\left(\frac{\partial u}{\partial \hat{x}^0}(\hat{x}^0, \ldots, x^n) \;\middle|\; x^0\right) + \frac{\partial}{\partial x^0} E\left(u(\hat{x}^0, \ldots, x^n) \;\middle|\; x^0\right) \\
&\ge E\left(\frac{\partial u}{\partial \hat{x}^0}(\hat{x}^0, \ldots, x^n) \;\middle|\; x^0\right) \\
&> E\left(\left(v(x^{(1)}, a, x^{(1)} \ldots, x^{(1)})\right)^2 \;\middle|\; x^0\right)^{1/2} \\
&\quad \times E\left(\left(\frac{\partial \log \zeta}{\partial \hat{x}^0}(x^1, \ldots, x^n \mid \hat{x}^0)\right)^2 \;\middle|\; x^0\right)^{1/2} \\
&\ge E\left[\left(p(x^1, \ldots, x^n) - D\right) \frac{\partial \log \zeta}{\partial \hat{x}^0}(x^1, \ldots, x^n \mid \hat{x}^0) \;\middle|\; x^0\right] \\
&= \frac{\partial P^M(x^0)}{\partial x^0},
\end{aligned}$$

where the first inequality follows because u is increasing and the signals affiliated, the second follows from (6), and the third follows from the Cauchy-Schwartz inequality and the fact that

$$p(x^1, \ldots, x^n, [\underline{x}, a]) < v(x^{(1)}, \overline{x}, x^{(1)}, \ldots, x^{(1)})$$

for the three mechanisms considered.

□

Lemma 4 $F_{Y_1}(\,\cdot\,|\,z, [\underline{x}, a])/f_{Y_1}(\cdot|z, [\underline{x}, a])$ *is decreasing in* a.

Proof: Use the notation $f_{Y_1,x^0}(\,\cdot\,,\,\cdot\,|\,z)$ for the joint probability density function of the signals Y_1 and x^0 conditional on the signal of bidder 1, $x^1 = z$. Take $a' \leq a$ and $x' \leq x$, and integrate the affiliation inequality to obtain

$$\begin{aligned} &\int_{\underline{x}}^{a'} f_{Y_1,x^0}(x, u|z)\, du \int_{a'}^{a} f_{Y_1,x^0}(x', u|z)\, du \\ &\quad \leq \int_{\underline{x}}^{a'} f_{Y_1,x^0}(x', u|z)\, du \int_{a'}^{a} f_{Y_1,x^0}(x, u|z)\, du. \end{aligned}$$

By adding $\int_{\underline{x}}^{a'} f_{Y_1,x^0}(x, u|z)\, du \int_{\underline{x}}^{a} f_{Y_1,x^0}(x', u|z)\, du$ to both sides, we get

$$\frac{\int_{\underline{x}}^{a} f_{Y_1,x^0}(x', u|z)\, du}{\int_{\underline{x}}^{a} f_{Y_1,x^0}(x, u|z)\, du} \leq \frac{\int_{\underline{x}}^{a'} f_{Y_1,x^0}(x', u|z)\, du}{\int_{\underline{x}}^{a'} f_{Y_1,x^0}(x, u|z)\, du}.$$

Now integrate both sides over $x' \in [\underline{x}, x]$ to finish the proof.

□

Lemma 5 *Let* g *and* h *be differentiable functions for which* (i) $g(\underline{x}) \geq h(\underline{x})$ *and* (ii) $g(x) < h(x)$ *implies* $g'(x) \geq h'(x)$. *Then* $g(x) \geq h(x)$ *for all* $x \geq \underline{x}$.

Proof: This is Lemma 2 in Milgrom and Weber (1982).

□

Proof of Theorem 2:

We denote by $W^M(x, z, X)$ the conditional expected payment made by bidder 1 in auction mechanism M if (i) the other bidders follow their equiibrium strategies, (ii) bidder 1's estimate is z, (iii) he bids as if it were x, (iv) he wins, and (v) all bidders believe that the owner's signal lies in X. To prove inequality (7), we use the fact that $W^{II}(z, z, X) \geq W^I(z, z, X)$. (See MW's Theorem 15 and its proof.) Hence, (7) follows because

$$P^M(x^0) \quad = \quad E\left(W^M(x^1, x^1, X) \;\middle|\; x^1 > Y_1, x^0\right) \;,$$

for $M \in \{I, II\}$.

□

Proof of Theorem 3:

It follows from MW, that when the anticipated sale set is X, and $X_1 > Y_1$, then the price in the second price auctions is $v(Y_1, Y_1, X)$, where

$$v(x, X) = E(V^1 \;\big|\; X_1 = x, Y_1 = y, X_0 \in X),$$

and the price in an English auction is $w(Y_1, Y_1, (Y_2, \ldots, Y_n), X)$, where

$$w(x, z, X) = E(V^1 \;\big|\; X_1 = x, Y_1 = y, (Y_2, \ldots, Y_n) = z, X_0 \in X).$$

Let $X = [\underline{x}, a]$ and $x > y$, and consider the inequality

$$\begin{aligned} & v(y, X) \\ & \quad = \;\; E(w(Y_1, Y_1, (Y_2, \ldots, Y_n), X) \;\big|\; X_1 = y, Y_1 = y, X_0 \in X) \\ & \quad \leq \;\; E(w(Y_1, Y_1, (Y_2, \ldots, Y_n), X) \;\big|\; X_1 = x, Y_1 = y, X_0 = a). \end{aligned}$$

This implies that

$$
\begin{aligned}
P^{II}(a, X) &= E(v(Y_1, Y_1, X) \bigm| X_1 > Y_1, X_0 = a) \\
&\leq E(w(Y_1, Y_1, (Y_2, \ldots, Y_n), X) \bigm| X_1 > Y_1, X_0 = a) \\
&= P^E(a, X).
\end{aligned}
$$

□

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