Finance in a Time of Disruptive Growth

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Abstract

We propose a unified theory of asset price determination and the dynamics of wealth inequality, with an emphasis on the pricing of "conventional" and "alternative" asset classes (private equity, real estate, etc.). Old firms are constantly displaced by young firms. The main feature of the model is that only a negligibly small set of arriving firms is able to capture the entire rents of innovation, which is the main driver of wealth inequality. We use the model to reconcile some seemingly contradictory patterns in private- vs. public-equity expected returns and examine the theoretical validity of some popular approaches to the risk-adjusting the returns of alternative assets. The model reproduces the fact that the wealth dynamics of the ultra-rich exhibit strong positive skewness, especially in the early stages of their accession to wealth. We also show how increased displacement risk naturally leads to the growth of alternative asset classes.

1 Introduction

Entrepreneurial risk and, more broadly, equity participation in non-publicly traded corporations are many times the origin of spectacular accession to wealth. Indeed, the returns on private equity positions are generally recognized as a leading driver behind the wealth dynamics of the ultra-rich. The allure of high returns in the sphere of non-publicly traded firms has also been the driving force behind the growth of private equity funds, which have become ever more important over the last four decades. Their premise is that the difficulty of access to privately-held investments creates some obvious gains from trade: The owners of privately held firms obtain access to capital and the ability to diversify their holdings ("risk sharing"), while the fund investors gain access to superior investment opportunities that are not available in public-equity markets.

We propose a neoclassical model to study these gains from trade.[1] The model has implications for (a) asset pricing and (b) the wealth dynamics of the ultra-rich. In terms of asset pricing, the model contains joint predictions for the rates of return on riskless bonds, private-equity investments, the aggregate stock market index, and the cross-section of publicly-traded equities. (Simple variations of the baseline model contain additional implications for the pricing of production factors such as real estate or commodities.) As we explain in greater detail shortly, such a unified, risk-based theory of "conventional" and "alternative" asset classes faces some challenges, since the cross-sectional patterns observed in public-equity markets seem at odds with some of the return patterns in private-equity markets. To reconcile these seemingly contradictory patterns, the model relies on the notion that risk sharing is (endogenously) imperfect both across and within cohorts of investors. One of the key impediments to perfect risk sharing is that returns to new ventures are quite skewed, which implies that some idiosyncratic risk necessarily remains in each private equity portfolio. This skewness of private venture returns is also the key driver for the model's implications for the wealth dynamics of the ultra-rich. Specifically, the model can help explain the intriguing fact that the cohort of rich investors newly entering the list of the ultra-wealthy (say, Forbes 400) exhibit a within-cohort wealth dispersion roughly similar to that of the investors already in that list.

We next describe the model's assumptions and summarize its implications for asset pricing and the dynamics of inequality. The backdrop is an infinite-horizon, stochastic, discrete-time, perpetual-youth, general equilibrium model of creative destruction. The production of

[1] We use the word "neoclassical" to signal that all expected excess returns are exclusively a reflection of risk.

a final good requires labor and intermediate products. New lines of (intermediate) products, which are introduced in stochastic amounts, raise aggregate production, but also displace the demand for, and hence the profits generated by, old intermediate products. The ownership rights to the production of the new product lines are allocated either to existing publicly traded firms or to newly arriving agents. The allocation of blueprints to new agents is random, and highly skewed. A small number of these agents end up with the profitable product lines, while the rest receive worthless allocations. As a result, the newly arriving agents are eager to share the allocation risk with investors, by offering their firm's shares for sale. They are, however, subject to an agency friction that leads them to only sell a fraction of their firm. This transaction is facilitated by financial intermediaries ("private equity funds") who purchase a portfolio of the new-firm shares on behalf of investors. This diversification, however, is costly; in particular, each intermediary optimally invests only in a subset of new firms.

There are therefore two impediments to perfect risk sharing. At the inter-cohort level, the fraction retained by the newly arriving entrepreneurs is excessive. At the intra-cohort level, each investor obtains a different rate of return on their private equity portfolio, depending on the subset of private firms she invests in. Inside the model this is the most important difference between investments in public and private equity: When investors invest in public equities, there is no dispersion in their returns; however, there is dispersion in their private equity positions. In the next section, where we summarize some empirical facts underpinning the model, we show that this distinction is a property of the data. At a theoretical level, an attractive feature of the model is that it captures as special cases the essence of several models in the literature, including the perfect risk-sharing limit (Rubinstein (1976), Lucas (1978)), the Constantinides and Duffie (1996) model, and the OLG model of Gârleanu et al. (2012).

The key results of the model can be summarized as follows. First, the model provides a unified, risk-based explanation of the returns obtained in conventional and alternative asset classes. This is not a straightforward task, because some cross-sectional patterns in the public equity market appear inconsistent with private equity markets: On the one hand the value premium requires that the "growth options" embedded in the price of publicly-traded growth firms command a comparatively low risk premium. On the other hand, this explanation of the value premium would also seem to imply that investments in venture-capital funds, which are presumably growth-option intensive, should consistently underperform any portfolio of public equities. The intuition is that these dynamic, new firms will displace some of the older, established firms at some point. Accordingly, an investment in new ventures should act as a hedge for the stock market, and should command a rate of return lower than public equities. In the data there is a paucity of such evidence.[2]

The model reconciles a positive value premium for public equities with a private equity premium that can be positive or negative. While the displacement risk of old firms by young firms is a concern to the investor, she also recognizes that the subset of firms in her private equity portfolio might not be among the select (measure-zero) set of new firms that will end up displacing the old firms. Compared to public equities, investors' private equity portfolio returns exhibit substantial cross-sectional dispersion (both in the model and the data), which implies that some amount of idiosyncratic risk is retained in these portfolios. Therefore, investments in new ventures expose the investor to idiosyncratic risk, without necessarily being an effective hedge for displacement risk.

Second, the model can be used to study the theoretical properties of some popular methods to "risk-adjusting" and evaluating the "out-performance" of private equity investments. Most popular amongst those approaches is the public market equivalent approach (PME) of Kaplan and Schoar (2005), which involves discounting private equity cashflows by the cumulative returns of the stock market and then dividing the sum of the discounted values by the amount invested. Values above one are interpreted as an indication of risk-adjusted outperformance. We show that — regardless of parametric assumptions — the expected value of the PME has to exceed unity, despite the fact that the excess returns of all assets reflect exclusively compensation for risk. This property of the PME is actually quite general and independent of the specifics of the model. The core intuition involves the fact that the public equity, used as discount factor, does not capture perfectly a marginal investor's portfolio risk, and therefore carries additional risk that biases the measure upwards.

Third, the same key feature of the model that drives its asset pricing predictions also drives its ability to explain some intriguing properties of wealth dynamics. As we show in the next section, the ultra-rich investors that are added to the Forbes 400 over every five-year period exhibit a wealth dispersion similar to the ultra-rich investors already in the list. Phrased differently, new entrants don't just enter the distribution of the ultra-rich at its lower ranks, they enter at all ranks. This appears inconsistent with the notion that the wealth growth of these entrants follows a diffusion process, since then one would expect the new entrants to replace predominantly the individuals at the lower ranks of the existing

[2] Returns in venture capital produce returns that are roughly comparable to public equities; there is no obvious evidence of substantial underperformance. Moreover, if the returns of venture capital are compared to similar returns in the public equity market (for instance the returns of the "small growth" portfolio), then venture capital seems to outperform.

distribution of the rich. Our model can help account for this fact, since the wealth dispersion among the rich occurs predominantly in the early stages of their accession to wealth. More broadly, inequality in the model is primarily driven by the churn of rich investors, i.e., the replacement of old rich by new rich. The old rich don't exhibit spectacular wealth-growth rates, as we discuss in greater detail in the next section.

Fourth, the model can help address some of the broad trends in the financial industry over the last four decades. If displacement risk increases (either because aggregate displacement activity increases, or just because the distribution of the gains from innovation becomes more concentrated), then the size of the private equity industry grows. In addition, there is an increased precautionary savings demand, which (for plausible parameter assumptions) drives down the real rate. In addition, any natural resource (land, commodities), which is useful to both the old and the newly arriving firms becomes a natural hedge for displacement risk and its value increases.

The rest of the paper is structured as follows. In Section 2, we summarize several empirical facts that motivate the model's key assumptions and form the targets of our calibration. Section 3 lays out the model, while Section 4 provides its solution and implications. In Section 5 we present a few extensions, notably to capture other factors of production, and in Section 6 we calibrate the model and compare its quantitative implications to the data. Finally, Section 7 concludes.

1.1 Literature review

Our paper relates to several strands of the literature.

Methodologically, the paper belongs to the well-developed literature that uses macroeconomic models to price the cross section of returns, and especially the size and value premium.One of the early contributions to this literature is Gomes et al. (2003), which develops a general equilibrium version of Berk et al. (1999), while more recent contributions include Papanikolaou (2011), Gârleanu et al. (2012), Gârleanu et al. (2012), and Kogan et al. (2020). Opp (2019) presents a tractable, macroeconomic model with venture capital, but focuses on different issues than we do in this paper. One of the goals of our paper is to extend this literature to a wider set of asset classes, and provide a resolution to the seemingly inconsistent pricing of growth options across public and private equities. A key role in our paper is played by creative destruction and "displacement risk," as in Gârleanu et al. (2012) and Kogan et al. (2020). There is a small literature that studies the impact of entry and imperfect competition on asset prices. Indicative examples are Loualiche (2021), Corhay et al. (2020), Dou et al. (2021), and Bena et al. (2015). Gârleanu et al. (2012) can be construed as a special case of this paper, obtaining if the market for private equity is shut down. Similarly, relative to Kogan et al. (2020) we introduce a market where entrepreneurs can trade their equity stakes with existing investors.

The lack of both inter-cohort and intra-cohort risk sharing plays an important role for the pricing of risk in our paper. There are two large and developed strands of the literature in asset pricing that pursue the asset-pricing implications of imperfections in both intergenerational[3] and intra-generational risk sharing.[4] The paper closest to ours is Constantinides and Duffie (1996). Indeed, our framework includes the core feature of Constantinides and Duffie (1996), namely a stochastic discount factor that incorporates higher aggregate moments through their impact on idiosyncratic risks, even though we make entirely different endowment assumptions.[5]

The (positively) skewed distribution of blueprints plays an important role in our paper. The skewness of idiosyncratic shocks is also a key element of such papers as Schmidt (2015), Constantinides and Ghosh (2017), and Ai and Bhandari (2021), except that the emphasis is on the *negative* skewness of labor income. Toda and Walsh (2019) and Gomez (2017) focus on the interaction between inequality and asset returns, similar to this paper.

A distinguishing feature of our model (compared to the literature on risk-sharing imperfections) is that we model the financial industry as a vehicle that facilitates transfers both within and across generations of entrepreneurs (behind the "veil of ignorance" about which firms are likely to be profitable). Purely from a technical perspective, our approach of modeling the financial industry as a facilitator of risk sharing resembles Gârleanu et al. (2015). However, the model in the present paper is intertemporal, features lack of both intra- and inter-cohort risk sharing, the arrival new blueprints follows an extremely skewed distribution (a Gamma process, as opposed to the Brownian-bridge construction in Gârleanu et al. (2015)), there is aggregate risk, and the model is amenable to calibration because of the usage of standard, homothetic utilities.

[3]Indicative examples of asset pricing papers featuring lack of intergenerational risk sharing are Abel (2003), Krueger and Kubler (2006), Geanakoplos et al. (2004), Campbell and Nosbusch (2007), Storesletten et al. (2007), Constantinides et al. (2002), Gomes and Michaelides (2005), Gârleanu and Panageas (2015), Schneider (2022), Maurer (2017), Ehling et al. (2018), Farmer (2018), Gârleanu et al. (2012), Gârleanu and Panageas (2023), Gârleanu and Panageas (2021).

[4]The asset prcing literature featuring uninsurable idiosyncratic shocks is vast and we do not attempt to summarize it here. See Panageas (2020) is a recent survey.

[5]See Krueger and Lustig (2010) on the importance of endowment assumptions in Constantinides and Duffie (1996).

While the study of inequality is not the primary focus of this paper, our model's assumptions are consistent with some recent studies of wealth inequality. Similar to Gomez (2023), we emphasize the role of entry, displacement, and churn in top wealth shares. Similar to Gabaix et al. (2016), we emphasize the importance of positively skewed, jump-like, idiosyncratic shocks, except that in our framework the positive skewness affects predominantly the entering cohorts.

Our paper has implications for the evaluation of investments in alternative asset classes. Our result that the quite popular PME method of Kaplan and Schoar (2005) has an expectation larger than one is novel, to the best of our knowledge. The interesting aspect of this result is that we can sign the direction of the bias, and that the bias obtains despite our making the most favorable assumptions for the validity of the PME. [6] Our model suggests a simple alternative to the PME approach, which is implemented in Korteweg et al. (2023).

Our model abstracts from "illiquidity" when valuing alternative asset classes.[7] Indeed, illiquidity presents an alternative explanation for the seemingly inconsistent cross-sectional pricing of growth options across public and private equities. We chose to abstract from considerations of illiquidity, because (a) we wish to highlight the role of the large cross-sectional dispersion of returns in alternative asset classes and its implications for the risk-reward tradeoff and (b) it helps us better make the point that the PME method has an expectation larger than one even in a world where any excess return is exclusively a reward for risk (be it diversifiable or aggregate). We also note that the illiquidity of alternative asset classes can have ambiguous effects on their prices: While some investors fret illiquidity, some long-term institutional investors who are subject to regulatory capital requirements and face convex costs of raising external funds (such as insurance companies), may be attracted by the fact that the net asset valuations of private equity are not as volatile as those of positions that need to be constantly marked to market. Additionally, over the course of the last decade the market for "secondaries" increased quite dramatically, and has substantially changed limited partners' ability to resale their private equity positions.

2 Empirical Motivation

Before presenting the model, we provide more detail on three empirical facts that we mentioned in the introduction. We focus on these facts because they are relatively novel and,

[6] Korteweg and Nagel (2016), Sorensen et al. (2014), Gupta and Van Nieuwerburgh (2021), and Korteweg et al. (2023) present alternatives to the PME approach.

[7] See, e.g., Ang et al. (2014).

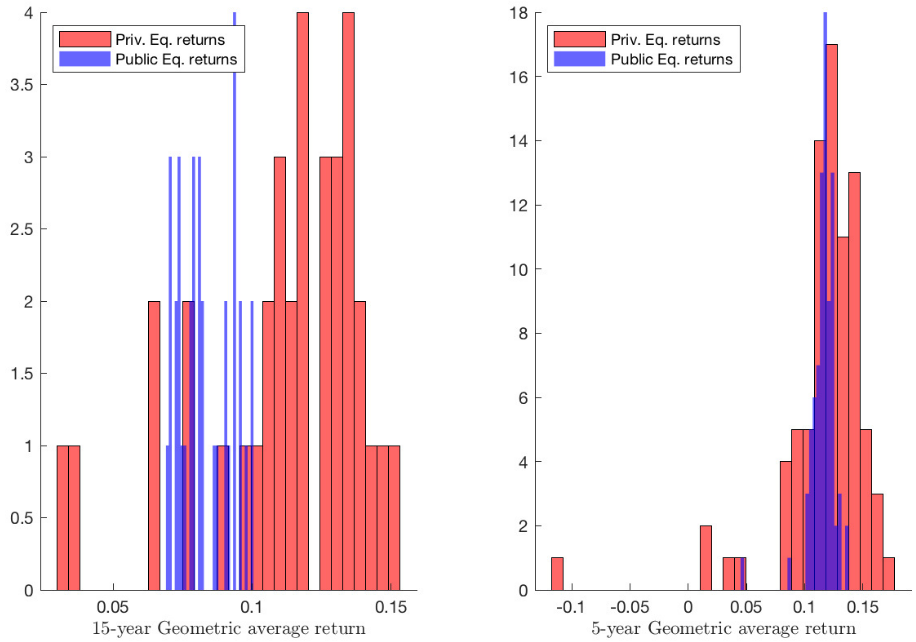

Figure 1: Return dispersion of the investments of public pension plans in public and private equity. Left plot: 15-year geometric averages (2003-2018). Right plot: five-year geometric averages (2013-2018).

most important, speak to our model's core assumptions. First, we document that the private equity returns tend to be far more disperse (across investors) than public equity returns. Second, we show that the wealth dynamics of new entrants into the population of the ultra-rich appear to exhibit jump-like features. Third, we document that the wealth dynamics of a fixed cohort of existing rich investors roughly align with the S&P 500 returns minus 2% per annum.

To establish the first fact, we examine the returns reported by public pension plans on their public equity and their private equity investments. These institutions constitute an attractive source of information, since they are required to file comprehensive annual financial reports ("CFAR") about the performance of each of their investments and are subject to "Freedom of Information Act" requests.[8]

Figure 1 shows a histogram of the returns obtained by public pension funds on their public (blue bars) and their alternative (red bars) equity investments. To mitigate concerns

[8]The source of the data is "Public Plans Data. 2001-2022," Center for Retirement Research at Boston College, MissionSquare Research Institute, National Association of State Retirement Administrators, and Government Finance Officers Association.

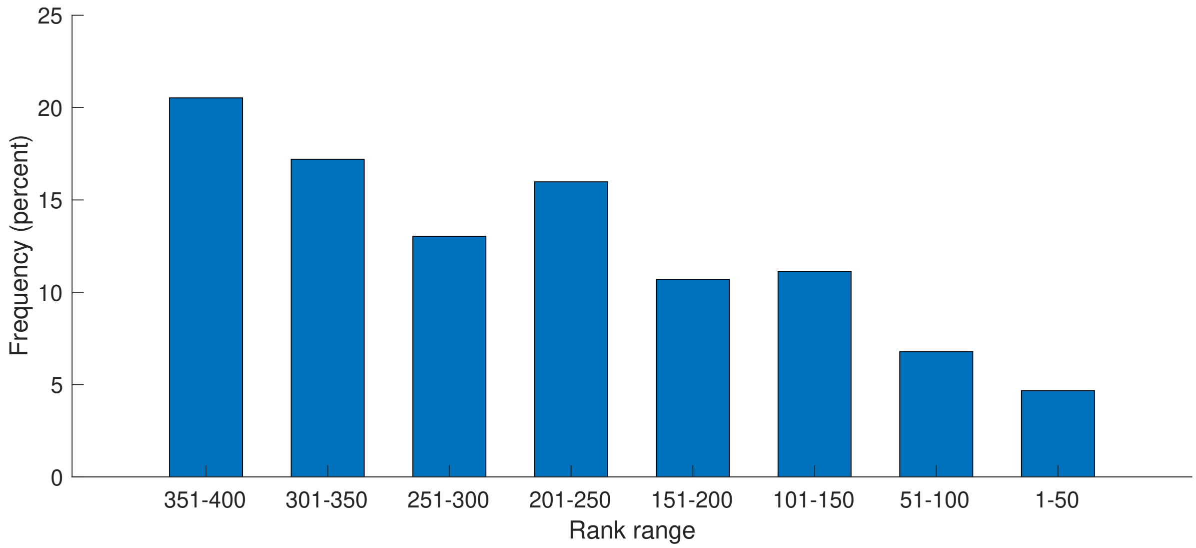

Figure 2: Proportion of self-made billionaires entering the Forbes 400 list in any of eight equal rank buckets.

that our results could be driven by the absence of mark-to-market values for the net asset values of unrealized exits, we compute a (geometric) average of the returns in both asset classes over periods of five (right plot) and 15 years (left plot).

The figure shows that the returns obtained by these large institutional investors in their public-equities investments is not very disperse. However, there is substantial dispersion in the returns that these investors obtain in their private-equity portfolios. This illustrates a key distinction between public and private equity that we want to capture in our model: investors obtain the same returns in their public equity investments, but their private equity returns can differ substantially.

Since our model contains implications for the dynamics of the ultra-rich, the second fact we document pertains to the dynamics of entrants into the Forbes 400 list. Forbes follows more than one thousand individuals and reports the 400 wealthiest ones each year. Figure 2 reports a remarkable feature concerning the entry of rich individuals into this list.

The figure plots the empirical distribution of the ranks of these entrants in the Forbes 400 list upon their entry. Since in our model calibration (Section 6) we use 5-year periods, we consider five-year windows starting in each of the years 1999-2014 and identify the new entrants into the list in each of these five-year periods. We use only entrants tagged by Forbes as “self made.” The remarkable feature of Figure 2 for our purposes is the depth of penetration of new entrants even in the right-most tail of the distribution. The individuals that just entered the top 400 don’t predominantly occupy the 350-400 ranks, but rather

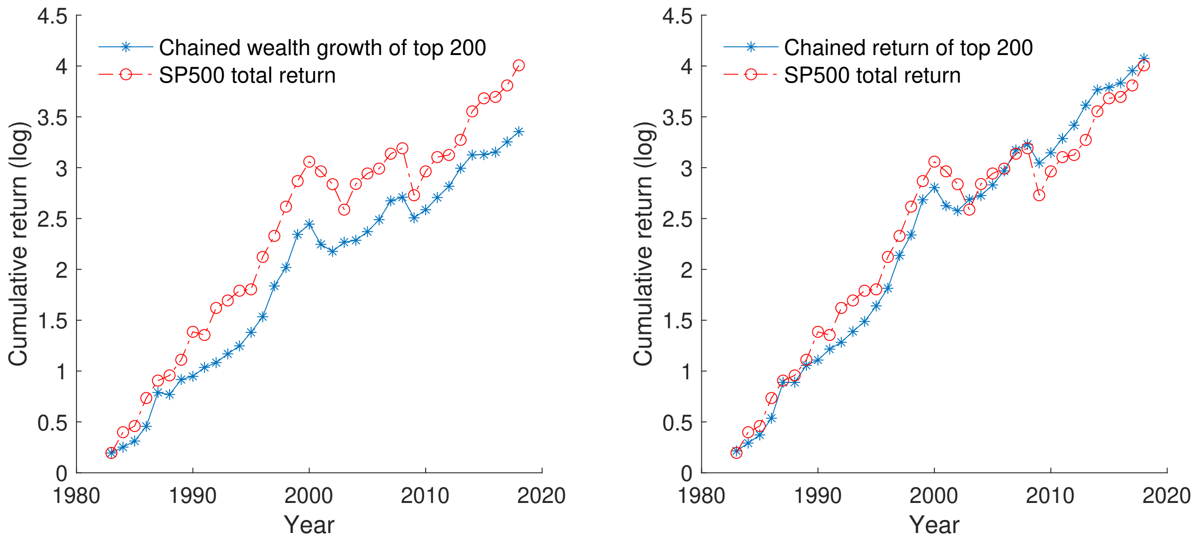

Figure 3: Left panel: Chained wealth growth of the top-200 group of individuals in the Forbes 400 list and S&P 500 total return. Right panel: Cumulative chained return on wealth to top-200 individuals under the assumption of a consumption-to-wealth ratio of 2% and S&P 500 total return. To compute wealth growth and return on wealth we divide the average time-$t+1$ wealth of time-t top-200 individuals still in the Forbes 400 list at time $t+1$ by the average time-$t+1$ wealth of all time-t top-200 individuals.

reach even the high ranks of the distribution of the existing rich. For instance, only 20.5% find themselves in the 350-400 range, while 22.6% populate the ranks 150 and above. As for the total number, the new entrants over the previous five years constitute approximately 20% of the composition of the list.

This penetration of new entrants into all ranks of the Forbes 400 distribution is consistent with the extreme skewness assumptions that underlie our model and forms one of the targets of our calibration.

The final motivating fact pertains to the average wealth dynamics of the existing rich. Specifically, the average return obtained by the extremely rich is quite similar to the S&P 500 total return, as illustrated by Figure 3. This figure graphs, with a continuous line, a measure of return on wealth computed as follows. Every year t, we take the top-200 individuals in the list[9] and compute their average wealth in year t as well as, for the ones still on the list in year $t+1$, their average wealth in year $t+1$. The return on wealth between t and $t+1$ is then approximated as the ratio of these two quantities plus an estimate of the consumption to wealth ratio, taken to be 2%. The right panel of the figure plots this return measure

[9]This group may include more than 200 names, due to ties.

together with the S&P500 return inclusive of dividends. The left panel simply plots the raw wealth growth next to the total index return.

To summarize, Figures 2 and 3 are consistent with the model's assumption that the rents from new firm creation are skewed and cause entry into the distribution of the ultra rich by new entrepreneurs (even into the highest ranks of this distribution). By contrast, the average return on wealth of the existing rich does not differ substantially from the returns of the assets already contained in the index.

3 Model

3.1 Agent preferences and demographics

We consider a model with discrete and infinite time: $t = \{..., 0, 1, 2, ...\}$. The size of the population is normalized to one. At each date a mass λ of agents are born, and a mass λ dies so that the population remains constant. We denote by $V_{t,s}$ the time-t utility of an agent born at time s. Investors have Epstein-Zin-Weil preferences with a unitary intertemporal elasticity of substitution and a risk aversion equal to γ:

$$\log V_{t,s} = (1 - \beta (1 - \lambda)) \log c_{t,s} + \beta (1 - \lambda) \log \mathcal{R}(V_{t+1,s}), \tag{1}$$

where $\mathcal{R}(x) = (\mathrm{E}_t x^{1-\gamma})^{\frac{1}{1-\gamma}}$ and $\beta \in (0, 1)$ is the agent's subjective discount factor, and $c_{t,s}$ is the agent's consumption at time t. Since the second term on the right hand side of (1) depends on the product of β and $(1 - \lambda)$, it is useful to define $\hat{\beta} = \beta(1 - \lambda)$ as the agent's effective discount factor.

3.2 Technology

As in Gârleanu et al. (2012), there is a continuum of intermediate-good firms that own non-perishable blueprints. Each blueprint allows the production of an intermediate good. The final good is produced by a representative, competitive firm, which purchases x_{jt} units of each intermediate good j and produces Y_t units of the final good. The production function is given by

$$Y_t = Z_t \int_0^{A_t} x_{jt}^{\alpha} dj$$

where Z_t is neutral productivity[10], A_t is the number of blueprints, and x_{jt} is the input of the intermediate input j at time t. The intermediate good j is supplied by firms engaging in monopolistic competition, and the production of one unit of the intermediate good j requires one unit of labor. Labor is supplied inelastically and the total measure of workers is $1 - \theta$.

Since this model of monopolistic competition is by now standard, we simply summarize the two results that are pertinent for our paper. First, total output, Y_t, is proportional to $Z_t A_t^{1-\alpha}$,

$$Y_t \propto Z_t A_t^{1-\alpha}. \tag{2}$$

Second, the profits accruing to any given blueprint at time-t are equal to

$$\pi_t = \frac{\alpha\left(1-\alpha\right) Y_t}{A_t} \propto A_t^{-\alpha}. \tag{3}$$

Equation (3) implies that the profit share of output is constant and equal to

$$A_t \pi_t = \alpha\left(1-\alpha\right) Y_t. \tag{4}$$

The remaining fraction, $(1 - \alpha(1 - \alpha))$, of output, Y_t, is the labor share in this economy.

Equations (2) and (3) reflect the fact that increasing the number of blueprints A_t raises total output Y_t (equation (2)), but the profits per blueprint decline (equation (3)). This is the sense in which this simple production specification captures the idea of displacement of old blueprints by new ones.

We would like to point out here that, even though we opted for a basic Romer-style production specification, the specific production assumptions (whether they are of the Romer type or the quality-ladder type) are irrelevant for the intuitions we develop in this paper.

3.3 New agents and products

The measure λ of newly born agents are of two types: a fraction θ are entrepreneurs and a fraction $1 - \theta$ are workers. Workers supply one unit of labor inelastically throughout their life. Since workers are not the focus of the paper, we assume that they are "hand-to-mouth" consumers, i.e., their wage income equals their consumption period by period. This assumption is not essential for the results, and we relax it in a later section.

Each period a total mass

$$\Delta A_{t+1} = A_{t+1} - A_t = \eta A_t \Gamma_{t+1} = \eta A_t \left(\Gamma_{t+1}^N + \Gamma_{t+1}^E + \Gamma_{t+1}^U\right) \tag{5}$$

[10] By "neutral" we mean productivity that does not lead to disruption of the profit shares of existing firms, as we explain below.

of new blueprints arrives. The proportional increment in A_t is captured by the random variable Γ_{t+1} and consists of three components, Γ^E_{t+1}, Γ^N_{t+1}, and Γ^U_{t+1}. Here, Γ^l_{t+1}, $l \in \{N, E, U\}$ are independent gamma distributed variables with shape parameters a^l and rate parameters b^l, and η is a constant. Since we want to focus on the asset-pricing implications of the model, we simplify matters and assume that new blueprints arrive exogenously. The new blueprints in the amount of $\eta A_t \Gamma^E_{t+1}$ accrue to existing firms, according to some distribution.[11] The blueprints $\eta A_t \Gamma^U_{t+1}$ accrue to newly arriving agents ("entrepreneurs"). The blueprints $\eta A_t \Gamma^N_{t+1}$ are also allocated to newly born entrepreneurs, except that the entrepreneurs who have the property rights to these blueprints are approached by private equity funds who offer to buy an equity share in those blueprints (we provide more details on the functioning of the private equity market below).

The crucial aspect for our analysis is that the new blueprints are allocated randomly to the newly arriving entrepreneurs. The entrepreneurs who have access to private equity funds are indexed by a "location" $i \in [0, 1)$ on a circle with circumference normalized to one. The number of blueprints distributed to entrepreneurs in location i is $\eta A_t d\Gamma^N_{i,t+1}$, where $d\Gamma^N_{i,t+1}$ denotes the independent increments of a gamma process, so that $\int_0^1 d\Gamma^N_{i,t+1} = \Gamma^N_{t+1}$.[12] Since the gamma process is not commonly used in economics, we summarize briefly some of its properties. To build intuition, we consider a discrete construction. We split the interval $[0, 1]$ into N equal intervals, and think of the gamma process at the location $\frac{k}{N}$ as a sum of gamma-distributed increments $\xi_{\frac{n}{N}}$,

$$\sum_{n=1}^{k} \xi_{\frac{n}{N}}, \tag{6}$$

where the pdf of the increment ξ_i is given by the gamma distribution

$$Pr(\xi_i \in dx) = \frac{b^{\frac{a}{N}}}{\Gamma\left(\frac{a}{N}\right)} x^{\frac{a}{N}-1} e^{-bx}\, dx. \tag{7}$$

The parameters $\frac{a}{N}$ and b are sometimes refered to as the "shape" and the "rate" of the gamma distribution, and $\Gamma\left(\frac{a}{N}\right)$ is the gamma function evaluated at $\frac{a}{N}$. The increments ξ_i are indepenent of each other, and the properties of the gamma distribution imply that

$$Pr\left(\sum_{n=1}^{N} \xi_{\frac{n}{N}} \in dx\right) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}\, dx, \tag{8}$$

[11] Given our assumptions of frictionless trading of existing firms, the distribution of these blueprints to existing firms is irrelevant. Since the representative investor holds all existing firms, the stochastic discount factor is affected only by the total number of blueprints allocated to existing firms, not the distribution.

[12] For technical reasons, we think of new entrepreneurs as indexed by $(i, j) \in [0, 1) \times [0, 1]$, with, for all j, (i, j) assigned to location i and receiving the same number of blueprints $\eta A_t d\Gamma^N_{i,t+1}$.

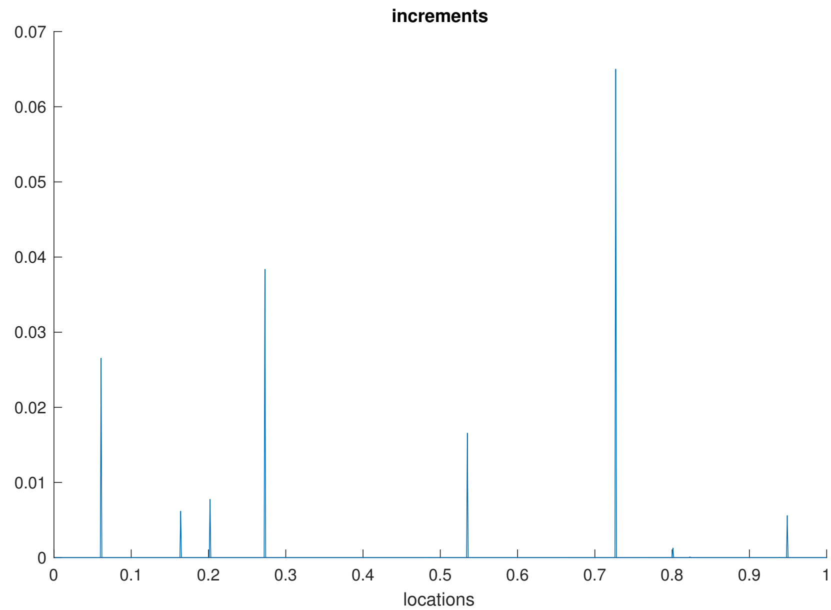

Figure 4: An illustration of the increments ξ_i, for the case $N = 1000$, $a = 1$, and $b = 2$.

which is the distribution of a gamma variable with shape a and rate b.

Using the gamma process is technically attractive for our purposes, since it captures in a stylized way the notion that entrepreneurship is very risky. This is illustrated in Figure 4. The figure shows a sample of increments ξ_i for the case where the interval $[0, 1]$ is split into $N = 1000$ subintervals. The figure illustrates that these increments tend to be essentially zero for almost all locations; however, a small subset of random locations exhibit big spikes of random height. From an economic point of view, this means that only the lucky few entrepreneurs who happen to find themselves in the locations exhibiting the large spikes obtain a non-trivial allocation of blueprints.

The limit of the variables given by (6) as the number of locations N goes to infinity is a gamma process. It is a positive and increasing process, whose paths are not continuous (they are only right continuous with left limits).[13] This implies that any given location is likely to receive a negligible allocation of blueprints. However, because the process $\Gamma^N_{i,t+1}$ is a discontinuous function of i, a zero-measure of locations receive a strictly positive measure of blueprints, and the entrepreneurs who find themselves in these locations become spectacularly wealthy.[14]

[13]The gamma process is, however, continuous in probability. This means that, for any $\varepsilon > 0$, $\lim_{\delta \to 0} \Pr(|\Gamma^N_{i+\delta,t+1} - \Gamma^N_{i,t+1}| > \varepsilon) = 0$.

[14]A similar outcome obtains concerning the untraded blueprints.

Before proceeding, we would like to note that this extreme-inequality setup is mostly for illustrative purposes and technical convenience. Less extreme distributions[15] would not affect the economic insights, as long as we preserve some notion of distributional risk.

A final crucial assumption is that no agent knows at time t the realization of the path of the gamma process Γ^N_{t+1}. Everyone is trading behind the "veil of ignorance" regarding which locations on the circle will obtain the valuable blueprints and which ones will obtain the useless ones. Newly arriving entrepreneurs are therefore eager to share that risk by selling shares to private-equity funds before this uncertainty is resolved. These shares entitle private-equity investors to a fraction υ of the profits that will be produced by the newly arriving firms in perpetuity. A fraction $1-\upsilon$ is "inalienable," a reduced form way of capturing incentive effects of equity retention.

To complete the description of the setup, we note that we have not yet specified how the blueprints $\eta A_t \Gamma^U_{t+1}$ are allocated to the entrepreneurs who don't have access to private equity funds. (Equivalently, one can think of these entrepreneurs as having in principle access to private equity funds, but with $\upsilon = 0$.) It turns out that for the purposes of solving for an equilibrium, the distribution of $\eta A_t \Gamma^U_{t+1}$ across the newly-born entrepreneurs is irrelevant. Only the total size of $\eta A_t \Gamma^U_{t+1}$ matters.[16]

3.4 Markets

At each point in time, an investor can trade a zero net-supply bond. To complete the market with respect to the random death events, we follow Blanchard (1985) and assume that agents can also trade annuities with competitive insurance companies that break even. These annuity contracts entitle an insurance company to collect the wealth W^j_t of an agent j in the event that she dies at time t and in exchange provide her with an income stream λW^j_t while she is alive. We refer to Blanchard (1985) for further details.

Investors at time t can trade costlessly in the shares of all companies created prior to time t. All such companies make the same profits π per blueprint. For future use, we denote

[15]Something as simple as assuming that locations are finite rather than a continuum would produce less extreme distributions without affecting the economic intuitions.

[16]However, this distribution matters when we discuss model implications for the wealth distribution. We postpone this discussion until we discuss the model's implications for the wealth distribution in the calibration section.

by Π_t the value of the future stream of profits from the representative blueprint:

$$\Pi_t = \mathrm{E}_t \left[\sum_{t+1}^{\infty} \frac{M_s^i}{M_t^i} \pi_s \prod_{t+1}^{s} (1 + \eta \Gamma_u^E) \right], \tag{9}$$

with M_s^i the marginal-utility process of a given investor. Note that future profits are due in part to the creation of new blueprints by existing firms.

Aside from shares in existing companies, investors can also buy shares of portfolios sold by private equity funds. Each private equity fund is positioned in a location i on the circle. This fund approaches the entrepreneurs in an arc of length Δ_i centered at location i, and purchases an equally weighted portfolio of cash-flow rights to the entrepeneus' blueprints. (Recall that these blueprints will arrive in period $t+1$, and their number is unknown at time t.) The private equity fund then offers the portfolio for purchase to investors. (Investors are entrepreneurs from previous periods). Each investor is randomly allocated to a position on the circle and purchases the portfolio offered by the private equity fund positioned in that location.

To prevent full diversification across all locations in the circle, we assume that diversification is costly. Specifically, when choosing an arc of length Δ_i, private equity funds expend resources equal to $\psi Y_t f(\Delta_i)$. (We make the cost proportional to Y_t to ensure that the cost of intermediation is a stationary fraction of the size of the aggregate economy.) Hence, to break even, the private equity fund needs to sell its portfolio for at least $\frac{1}{\Delta_i} \int_{i-\frac{\Delta_i}{2}}^{i+\frac{\Delta_i}{2}} P_t^j dj + \psi Y_t f(\Delta_i)$, where P_t^j is the price that the private equity fund must pay to the entrepreneur in location j for purchasing an equity share to the blueprints she sells. Assuming that there exist location-invariant equilibria, such that $P_t^j = P_t$, the price of the portfolio is simply $P_t + \psi Y_t f(\Delta_i)$. To simplify notation, from now on, we guess that there exist equilibria with $P_t^j = P_t$, and will then verify their existence in the next section. Likewise, we conjecture that Δ_i is independent of i, and write $\Delta_i = \Delta$. For further convenience, we let $\hat{P}_t = P_t + \psi Y_t f(\Delta)$.

Figures 5 and 6 illustrate how private equity funds can facilitate risk sharing in this economy. By purchasing an equal-weighted portfolio of shares on an arc of length Δ, the private equity funds are able to "smooth out" the spikes of the gamma process. Indeed, as the figure illustrates, they can offer their investors a portfolio of blueprints that has the same mean as the number of blueprints that arrive in each location, but is second-order stochastically dominant. Specifically, by using properties of the gamma distribution, one can show that $\frac{1}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j^N$ is gamma distributed with shape $a^N \Delta$ and rate $b^N \Delta$, and accordingly it has mean equal to $\frac{a^N}{b^N}$ and standard deviation equal to $\frac{\sqrt{a^N}}{b^N \sqrt{\Delta}}$. Further, if $\Delta_2 > \Delta_1$, then

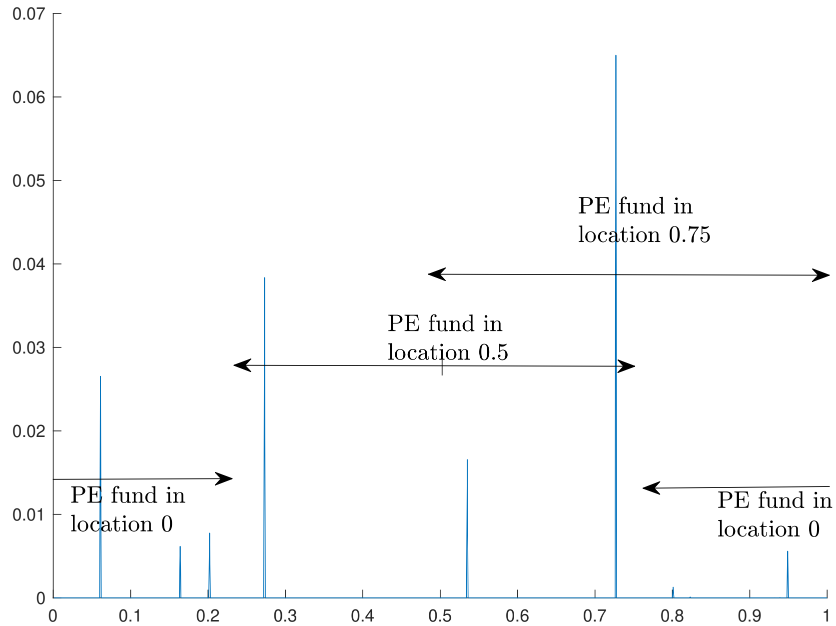

Figure 5: An illustration of how private equity funds help with risk sharing. The increments are the same as in Figure 4, and $\Delta = 0.5$. The private equity fund in position 0.5 provides an equal weighted portfolio of the increments in [0.25, 075]. The private equity fund in position 0.75 averages the increments in [0.5, 1], while the private equity fund in position 0 averages the increments in $[0, 0.25] \cup [0.75, 1]$.

$\frac{1}{\Delta_2} \int_{i-\frac{\Delta_2}{2}}^{i+\frac{\Delta_2}{2}} d\Gamma_j^N$ second-order stochastically dominates $\frac{1}{\Delta_1} \int_{i-\frac{\Delta_1}{2}}^{i+\frac{\Delta_1}{2}} d\Gamma_j^N$. Fur future reference, we note that the correlation between the blueprints accruing to a given fund $\frac{1}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j^N$ and total new blueprints accruing to private equity funds, $\int_0^1 d\Gamma_j^N$, is $\sqrt{\Delta}$. Given the location invariance of the set-up, this correlation does not depend on i.

Private equity funds in each location are competitive and in an effort to attract investors they determine Δ in a way that maximizes investor welfare. Moreover, the assumption of perfect competition ensures that private equity funds make no profits in equilibrium. Accordingly, they act as simple pass through entities. They enable a risk-sharing agreement between investors and entrepreneurs, by creating a market for early access to blueprints.

To ensure heterogeneity in the returns of existing investors, we assume that $f(1) = \infty$, implying that Δ lies in the interior of $[0, 1]$.

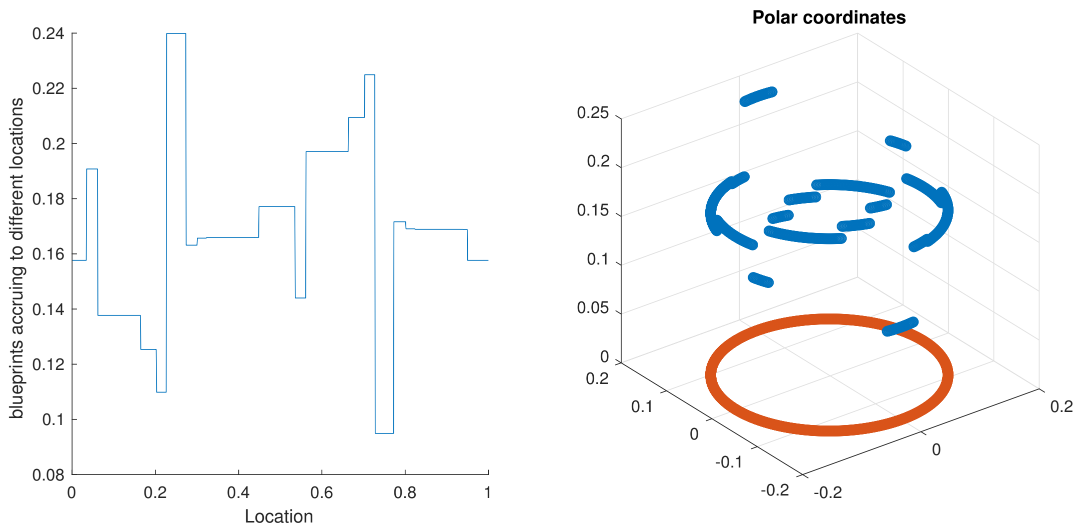

Figure 6: The distribution of equal weighted returns. The left figure depicts the blueprints accruing to the portfolio formed by the private equity fund in each location i, which is simply an equal weighted average of the blueprints accruing to locations in an arc Δ around the private equity fund's location. The right figure is identical to the left figure except that the results are now depicted in polar coordinates.

3.5 Budget constraints

With these assumptions, the dynamic budget constraint of an investor who resides in location i can be expressed as

$$W_t^i = S_t^{E,i} A_t \Pi_t + B_t^i + S_t^{N,i} \hat{P}_t + c_t^i, \tag{10}$$

$$W_{t+1}^i = S_t^{E,i} A_t \left(\Pi_{t+1} + \pi_{t+1}\right) + (1 + r_t^f) B_t^i + \upsilon S_t^{N,i} A_t \left(\Pi_{t+1} + \pi_{t+1}\right) \frac{\eta}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}^N + \lambda W_{t+1}^i,$$

where $S_t^{E,i}$ is the number of shares of the (representative) firm that is already traded at time t, B_t^i is the amount invested in bonds, r_t^f the interest rate, and $S_t^{N,i}$ is the number of shares purchased in the private equity fund in location i. (We note that W_{t+1}^i is the agent's wealth at $t+1$ conditional on survival and the term λW_{t+1}^i in the second line of (10) represents annuity income.) We normalize the supply of shares of all firms to unity. A convenient way to express (10) is

$$\frac{W_{t+1}^i}{W_t^i} = \frac{1 - \frac{c_t^i}{W_t^i}}{1 - \lambda} \left(\phi_B^i \left(1 + r_t^f\right) + \phi_E^i R_{t+1}^E + \phi_N^i R_{t+1}^{N,i}\right), \tag{11}$$

where $\phi_B^i \equiv \frac{B_t^i}{W_t^i - c_t^i}$, $\phi_E^i \equiv \frac{S_t^{E,i} A_t \Pi_t}{W_t^i - c_t^i}$, and $\phi_N^i = \frac{S_t^{N,i} \hat{P}_t}{W_t^i - c_t^i}$ are the post-consumption wealth shares invested by investor i in bonds, existing firms, and newly arriving firms respectively, and $R_{t+1}^E \equiv \frac{\Pi_{t+1} + \pi_{t+1}}{\Pi_t}$ and $R_{t+1}^{N,i} \equiv \frac{A_t}{\hat{P}_t} \frac{\eta \upsilon}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} (\Pi_{t+1} + \pi_{t+1}) \, d\Gamma_{i,t+1}^N$ are the gross returns of existing firms and the portfolio of newly arriving firms that investor i invests in.

An important observation about (11) is that, as long as $P_t^j = P_t$ for all $j \in [0, 1)$, the portfolio choices ϕ_B^i, ϕ_E^i, and ϕ_N^i, as well as the choices of Δ_i and of $\frac{c_t^i}{W_t^i}$ are the same for all investors, irrespective of their level of wealth and the location where they reside at time t, which simplifies the solution and analysis of the model.

To ensure that $P_t^j = P_t$ for all $j \in [0, 1)$ we make one final, "free entry" assumption: investors re-locate prior to the start of trading in each period, so that the total wealth of all the investors positioned in each location becomes equal across locations.[17]

Conditional on all private equity funds choosing the same arc-length $\Delta_i = \Delta$, and given the location-invariant nature of the distribution of new firms across the circle, the investors have the incentive to position themselves in locations that offer lower prices for a share to the portfolio of new firms. This free investor movement equalizes prices across locations and occurs when the wealth of investors positioned in every location is identical across locations.

3.6 Location-invariant equilibrium

The definition of a location-invariant equilibrium is standard. Such an equilibrium is a collection of prices Π_t and P_t, portfolio allocations ϕ_B, ϕ_E, and ϕ_N, a choice of Δ, and consumption processes for all agents c_t^j such that a) given prices, ϕ_B, ϕ_E, ϕ_N, Δ, and c_t^j are choices that maximize (1) subject to (11), b) the consumption market clears: $\int_j dc_t^j = A_t \pi_t - \psi Z_t A_t^{1-\alpha} f(\Delta)$, c) the markets for all shares (both new and existing) clear: $\int_j dS_t^{E,j} = \int_j dS_t^{N,j} = 1$, and d) the bond market clears: $\int_j dB_t^j = 0$.

4 Solution

Next we construct an equilibrium that is both location-invariant, time-invariant and symmetric, in the sense that all agents choose the same portfolio. Specifically, we conjecture that there exists an equilibrium whereby $\phi_B = 0$, and the portfolio shares ϕ_E and ϕ_N, the interest rate r^f, the participation arc Δ, the valuation ratios $P^E \equiv \frac{\Pi_t}{\pi_t}$, $P^N \equiv \frac{P_t}{A_t \pi_t}$, and $\hat{P}^N \equiv \frac{\hat{P}_t}{A_t \pi_t}$, and the consumption-to-wealth ratio $c \equiv \frac{c_t^i}{W_t^i}$ are the same for all agents and constant across

[17] Mathematically, such a re-location is always possible; one of the infinitely many ways to achieve it is to assign the investor with wealth W_t^j to location $F^{-1}(W_t^j)$, where $F(\cdot)$ is the wealth distribution.

time. After computing explicit values for the constants that support such an equilibrium, we provide sufficient conditions for its existence.

For the remainder of this section we specialize the model to the case of logarithmic preferences. Later we show how to extend the results to the case of recursive preferences with general risk aversion, a case that we also use as basis for our quantitative evaluation.

Maintaining the supposition that the price-to-earnings ratio for existing firms $\frac{\Pi_t}{\pi_t}$ is a constant, denoted by P^E, the return R^E_{t+1} can be expressed as

$$R^E_{t+1} = \frac{\pi_{t+1} + \Pi_{t+1}}{\Pi_t}\left(1 + \eta\Gamma^E_{t+1}\right) = \frac{\pi_{t+1}}{\pi_t}\left(\frac{1+P^E}{P^E}\right)\left(1 + \eta\Gamma^E_{t+1}\right) \tag{12}$$

$$= \frac{Z_{t+1}}{Z_t}\left(\frac{1+P^E}{P^E}\right)\left(\frac{A_{t+1}}{A_t}\right)^{-\alpha}\left(1 + \eta\Gamma^E_{t+1}\right) \tag{13}$$

$$= \frac{Z_{t+1}}{Z_t}\left(\frac{1+P^E}{P^E}\right)\left(1 + \eta\Gamma_{t+1}\right)^{-\alpha}\left(1 + \eta\Gamma^E_{t+1}\right). \tag{14}$$

Moreover, using similar reasoning, the return on a private-equity investment is

$$R^N_{i,t+1} = \frac{\upsilon A_t\left(\Pi_{t+1} + \pi_{t+1}\right)\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1}}{A_t \pi_t \hat{P}^N} \tag{15}$$

$$= \frac{R^E_{t+1}}{1+\eta\Gamma^E_{t+1}}\frac{P^E}{\hat{P}^N}\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t+1} \tag{16}$$

$$= \frac{P^E}{\hat{P}^N}R^E_{t+1}H_{i,t+1}, \tag{17}$$

where we defined the variable $H_{i,t+1}$ that captures the wedge between the two returns:

$$H_{i,t} := \left(1 + \eta\Gamma^E_{t+1}\right)^{-1}\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma^N_{i,t}. \tag{18}$$

From here on we omit the dependence of the random variables R^N_{t+1} and H_{t+1} on i, since all the statements we make pertain to the joint distribution of this variable and aggregate quantities, which is independent of i.

The following proposition contains an explicit description of a symmetric time- and location-invariant equilibrium.

Proposition 1 *Assuming that a location-invariant, time-invariant, and symmetric equilib-*

rium exists, the unique equilibrium values of ϕ_B, ϕ_E, ϕ_N, and c are

$$\phi_B = 0 \tag{19}$$

$$\phi_E = \frac{E\left[\left(R_t^E\right)^{1-\gamma}(1+H_t)^{-\gamma}\right]}{E\left[\left(R_t^E\right)^{1-\gamma}(1+H_t)^{1-\gamma}\right]} \tag{20}$$

$$\phi_N = 1 - \phi_E \tag{21}$$

$$c = 1 - \hat{\beta}. \tag{22}$$

With $\delta = \frac{\psi Y_t}{A_t \pi_t}$, the equilibrium values of P^E and $\hat{P}^N$ are given by

$$P^E = \phi_E\left(1 - \delta f(\Delta)\right)\frac{\hat{\beta}}{1-\hat{\beta}}, \tag{23}$$

$$\hat{P}^N = \left(1 - \phi_E\right)\left(1 - \delta f(\Delta)\right)\frac{\hat{\beta}}{1-\hat{\beta}}, \tag{24}$$

and the interest rate equals

$$1 + r^f = \frac{E\left[\left(R_t^E\right)^{1-\gamma}(1+H_t)^{-\gamma}\right]}{E\left[\left(R_t^E\right)^{-\gamma}(1+H_t)^{-\gamma}\right]}. \tag{25}$$

Finally, the equilibrium value of Δ is given by the solution to the equation

$$\frac{1-\hat{\beta}}{\hat{\beta}}\frac{\delta f'(\Delta)}{1-\delta f(\Delta)} = \frac{\partial E\left[\log\left(1+H_t\right)\right]}{\partial \Delta} \tag{26}$$

if utility is logarithmic and to the equation

$$(\gamma - 1)\frac{1-\hat{\beta}}{\hat{\beta}}\frac{\delta f'(\Delta)}{1-\delta f(\Delta)} = -\frac{\partial}{\partial \Delta}\log\left(E\left[\left(R_t^E\right)^{1-\gamma}(1+H_t)^{1-\gamma}\right]\right) \tag{27}$$

otherwise, i.e., for $\gamma \neq 1$.

We analyze the properties of the equilibrium in steps. First, we derive the implications of the equilibrium for risk sharing both within and across cohorts of entrepreneurs. Then we present results on the equilibrium expected excess returns of existing firms and new ventures, and highlight implications for performance evaluation, in particular the PME measure. We also discuss implications for the size of the financial industry.

4.1 Risk-sharing implications

For presentation purposes, it is convenient to start the analysis by treating Δ not as a choice variable, but rather as a fixed parameter.

To derive the implications of the model for risk sharing between and across investor cohorts, we start with the following proposition.

Proposition 2 *Aggregate wealth growth is given by*

$$\frac{W_{t+1}}{W_t} = \frac{Z_{t+1}}{Z_t}\left(1+\eta\Gamma_{t+1}\right), \tag{28}$$

while an individual investor's wealth growth (conditional on survival) is given by

$$\frac{W_{t+1}^i}{W_t^i} = \frac{W_{t+1}}{W_t}\left(\frac{1}{1-\lambda}\right)\left(\frac{1+P^E}{1+P^E+P^N}\right)\left(\frac{1+\eta\Gamma_{t+1}^E+\eta\upsilon\Gamma_{t+1}^N}{1+\eta\Gamma_{t+1}}\right)X_{i,t+1}, \tag{29}$$

where

$$X_{i,t+1} \equiv \frac{1+\eta\Gamma_{t+1}^E+\eta\upsilon\Gamma_{t+1}^N\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}}{1+\eta\Gamma_{t+1}^E+\eta\upsilon\Gamma_{t+1}^N} \tag{30}$$

$$dL_{j,t+1} \equiv \frac{d\Gamma_{j,t+1}^N}{\Gamma_{t+1}^N}. \tag{31}$$

Equation (29) shows that risk is imperfectly shared both within and across cohorts. The impairment of within-cohort risk sharing is captured by the term $X_{i,t+1}$, which reflects heterogenous returns experienced by existing investors. This heterogeneity is driven by the investors' inability to invest in all available ventures. Indeed, when $\Delta = 1$ and investors can purchase rights to all blueprints across the circle, the term $X_{i,t+1}$ becomes one, and the within-cohort lack of risk sharing disappears.

However, risk-sharing is limited not only along the intra-cohort dimension, but also across cohorts. Even if $\Delta = 1$, equation (29) shows that individual wealth $\frac{W_{t+1}^i}{W_t^i}$ and aggregate wealth $\frac{W_{t+1}}{W_t}$ are not perfectly correlated as long as either (a) entrepreneurs have to retain some equity ($\upsilon < 1$) or (b) some blueprints accrue to entrepreneurs who have no access to the private equity funds ($\Gamma_{t+1}^U \neq 0$). The random term $\frac{1+\eta\Gamma_{t+1}^E+\eta\upsilon\Gamma_{t+1}^N}{1+\eta\Gamma_{t+1}^E+\eta\Gamma_{t+1}^N+\eta\Gamma_{t+1}^U}$ in equation (29) captures the inter-cohort lack of risk sharing. In general this term is random and smaller than one, except in the special case where $\upsilon = 1$ and $\Gamma_{t+1}^U \equiv 0$.

In summary, Δ controls the extent of intra-cohort risk sharing, while υ and Γ_{t+1}^U control inter-cohort risk sharing. If $\Delta = \upsilon = 1$, and all entrepreneurs have access to private equity ($\Gamma_{t+1}^U = 0$), then risk is perfectly shared both within and accross cohorts; individual wealth growth and aggegate wealth growth are perfectly correlated. However, even in that case individual and aggregate wealth-growth rates differ by a negative constant. Indeed, aggregating the wealth growth of all investors surviving into $t+1$, we obtain

$$\log\left((1-\lambda)\frac{\int_i W_{t+1}^i di}{W_t}\right) - \log\left(\frac{W_{t+1}}{W_t}\right) = \log\left(\frac{1+P^E}{1+P^E+P^N}\right) < 0. \tag{32}$$

The negative constant reflects the fact that the existing investors have to pay the arriving entrepreneurs to purchase their blueprints. If all blueprints accrued to existing firms ($\Gamma_{t+1}^N = \Gamma_{t+1}^U = 0$) then the entire portfolio of the investor is invested in existing firms ($\phi_E = 1, \phi_N = 0$); therefore $P^N = 0$ and aggregate and individual growth rates are identical.

4.2 Implications for the SDF

Since the wealth-to-consumption ratio is constant, our conclusions on wealth changes apply without modification to consumption changes of individual investors: an individual investor's consumption change is given by the right hand side of (29). The SDF M_t^i of an individual investor is given by

$$\begin{aligned} \frac{M_{t+1}^i}{M_t^i} &= \beta(1-\lambda)\left(\frac{W_{t+1}^i}{W_t^i}\right)^{-\gamma} \\ &\propto (1+\eta\Gamma_{t+1})^{\gamma\alpha}\left(1+\eta\Gamma_{t+1}^E + \frac{\eta\upsilon}{\Delta}\Gamma_{t+1}^N\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}\right)^{-\gamma}\left(\frac{Z_{t+1}}{Z_t}\right)^{-\gamma}. \end{aligned} \tag{33}$$

For the markets where all investors are participating (in particular, the market for existing stocks and the risk-free asset), any $\frac{M_{t+1}^i}{M_t^i}$ is a valid SDF, and so is

$$\frac{M_{t+1}}{M_t} \equiv \mathrm{E}_t\left[\frac{M_{t+1}^i}{M_t^i}\middle|\Gamma_{t+1}^E, \Gamma_{t+1}^N, \Gamma_{t+1}^U\right]. \tag{34}$$

By the properties of gamma distributed variables, the quantity $\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$ is beta distributed and independent of Γ_{t+1}. For our purposes, the interesting property of $\frac{M_{t+1}}{M_t}$ is its covariance with the growth shock Γ_{t+1}^N. To isolate this covariance, we assume that Z_t is non-stochastic. We obtain the following.

Proposition 3 *Assume that the neutral-shock process Z_t is deterministic for all t.*

(i) When risk sharing both across and within cohorts is perfect, i.e., $\Delta = 1$ and $\upsilon = 1$, the SDF $\frac{M_{t+1}}{M_t}$ covaries negatively with the innovation shocks Γ_{t+1}^N: Cov$\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right) < 0$.

(ii) By contrast, Cov$\left(\frac{M_{t+1}}{M_t}, \Gamma_{t+1}^N\right) > 0$ when either υ or Δ is sufficiently small.

It always holds that $Cov\left(\frac{M_{t+1}}{M_t}, \Gamma^U_{t+1}\right) > 0$.

Proposition 3 shows how risk sharing imperfections can determine whether the marginal utility of consumption of the representative investor rises or declines as the innovative activity by new entrants, Γ^N_{t+1}, increases. If risk is shared perfectly both within and across cohorts, then large realizations of Γ^N_{t+1} are "good news" for the representative investor. The gains in the value of the portfolio of new firms are enough to undo the reduction in the value of the existing assets owned by the investor (since the arriving firms capture some of the profits of existing assets). However, away from the perfect risk-sharing limit, large realizations of Γ^N_{t+1} are "bad news." For instance, if risk is shared perfectly within cohorts ($\Delta = 1$) but imperfectly across cohorts ($\upsilon < 1$), then a fraction of the value of new ventures cannot be separated from the newly arriving cohort of agents. Hence, the arrival of new blueprints leads to the portfolio of existing assets cannot be offset by the gains on the portfolio of new ventures. This is the key insight of Gârleanu et al. (2012).

Even if risk is perfectly shared across cohorts ($\upsilon = 1, \Gamma^U_{t+1} \equiv 0$), large realizations of Γ^N_{t+1} may be (unconditionally) perceived as states of high marginal utility (bad states) when Δ is sufficiently small. In this situation existing investors *as a group* gain from increased innovation, since they buy all the shares of the newly arriving entrepreneurs before the realization of Γ^N_{t+1}. However, the investors do not know ex ante whether they will receive a large or a small allotment of the new firms. Because of their risk aversion, they assign greater weight to the event that they end up with a disproportionately small share of the gains from growth, and therefore they perceive a high realization of Γ^N_{t+1} as bad news. This intuition is reminiscent of that put forth by Constantinides and Duffie (1996), Kogan et al. (2020), and Gârleanu et al. (2015).

Hence our model nests models of perfect risk sharing as well as of imperfect risk sharing, across and within cohorts, as special cases. However, the most important difference is that it proposes a view of the financial industry as a mechanism to improve risk sharing, yielding predictions on the joint determination of expected returns, the size of the financial industry, the interest rate, the wealth distribution, etc.

4.3 Equilibrium excess returns and public market equivalents

Having derived the equilibrium SDF, we can now discuss the implications of the model for the investment properties of the new ventures as an asset class. It is important to keep in mind throughout that there is no concept of skill or abnormal return in our model. We start by comparing their expected returns with those of existing firms. We then consider a

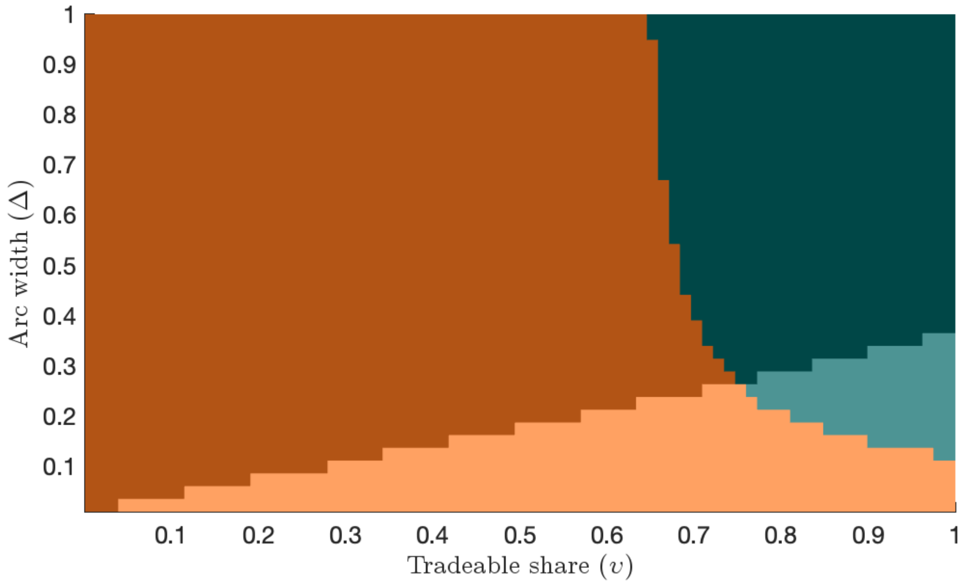

Figure 7: The figure shows, for each combination (υ, Δ) of new-venture sales by entrepreneurs and investment arcs, whether the covariance between marginal-utility growth M^i_{t+1}/M^i_t and innovation shock Γ^N is positive — colored brown — and whether $\mathrm{E}[R^{N,i} - R^E] > 0$ — light hues.

performance metric that is widely used in the alternative-asset space, namely public market equivalents (PME).

As we noted in the previous section, existing firms demand a risk premium. The innovation shocks accruing to new ventures (Γ^N), on the other hand, may covary either positively — when not shared sufficiently with existing investors, because either υ or Δ is small — or negatively with an investor's marginal utility. In this section we would like to highlight that the sign of this correlation, which we can regard as a value premium, neither determines nor is determined by the sign of the difference in returns between new ventures and existing firms, i.e., $\mathrm{E}[R^{N,i} - R^E]$.

To that end, we first state a formal result.

Proposition 4 *Let Z be independent of all other random variables and $\gamma \geq 1$.*

(i) Fixing $\Delta > 0$, for υ small enough it holds that $E[R^{N,i}] < E[R^E]$.

(ii) Fixing $\upsilon > 0$, if Δ is small enough then $E[R^{N,i}] > E[R^E]$.

The result in Proposition 4 that is most noteworthy is contained in part (ii). Combined with Proposition 3, it establishes that a negative risk premium demanded by the innovation factor Γ^N accruing to new ventures can coexist with a positive risk premium for new ventures, in fact even higher than for existing firms. The reason, hinted at by the condition that Δ be small, is the idiosyncratic risk that each new-venture portfolio carries.

We highlight the observation that a value premium need not imply lower returns for "growthy" (PE-backed) new ventures as being most novel and empirically relevant. Of course, the new-venture returns may be lower, as suggested by their growth status. In fact, they can be lower even when new ventures do not provide effective insurance against displacement because existing agents have adequate access to these ventures, and therefore are not displaced by them (only by the untraded ventures). In this case, new blueprints introduced by existing and new firms are qualitatively the same, and the two expected returns are determined by the cash-flow betas with respect to aggregate growth.

One implication of Proposition 4 we wish to emphasize is that new ventures may offer a higher return despite the absence of investment "skill" among fund managers and lower systematic risk. They key to this observation is the undiversifiable idiosyncratic risk that investors end up assuming in buying these assets. No correction for market risk — whether estimates of beta or variants of estimating "public market equivalents" — accounts for this additional risk.

We illustrate these observations graphically with Figure 7.[18] We use color coding to indicate simultaneously, for any combination (υ, Δ), whether Γ^N shocks hedge investors — colored brown, rather than green — and whether expected returns on new technologies, $\mathrm{E}[R^{N,i}]$ are higher than on existing firms, $\mathrm{E}[R^E]$ — light hues means that they are.

The southwest region of the figure illustrates Proposition 3(ii) — the brown color meaning that Γ^N provides a hedge to investors ($\mathrm{Cov}_t(M^i_{t+1}, \Gamma^N_{t+1}) > 0$). At the same time, while the left edge shows that, for a sufficiently small tradable share ν, $\mathrm{E}[R^{N,i}] < \mathrm{E}[R^E]$ as stated by Proposition 4(i), the light hues at the bottom of the chart encode 4(ii): when diversification across new ventures is sufficiently poor — Δ is small — new ventures return more than old ones. Naturally, the north-east corner corresponds to a complete market, in which Γ^N is an aggregate shock requiring a positive risk premium (Proposition 3 (i)).

In light of the discussion surrounding Proposition 4, it may come as no surprise that correcting for market returns by computing PME measures (Kaplan and Schoar, 2005) does not uncover abnormal returns, either. In fact, an even stronger result holds:

Proposition 5 *For any $\Delta \in [0,1]$ and any $\upsilon \in [0,1]$ it holds that $E\left[\frac{R_t^{N,i}}{R_t^E}\right] \geq 1$ if γ is equal (or sufficiently close) to 1.*

[18] We use the parameters detailed in Section 6, with the sole modification that the process Z_t is taken independent of all other variables.

This is a quite remarkable result, in particular since it holds precisely under the preference assumptions — logarithmic utility — that justify the PME measure. It is crucial, of course, that the market return used in the calculation does not capture the entire market, or the total return on wealth of any (marginal) investor. The result extends beyond the specifics of our model, and follows whenever the two returns in question combine to give an investor's overall portfolio return.

The argument is simple enough to sketch in a few lines: If $R_w = a_1 R_1 + a_2 R_2$ is the investor's total return, obtained by combining returns R_1 and R_2 with weights $a_1 < 1$ and a_2, $a_1 + a_2 = 1$, then the logarithmic utility means that the pricing kernel is proportional to R_w^{-1}, and we can write

$$1 = \mathrm{E}\left[\frac{R_w}{R_w}\right] = \mathrm{E}\left[\frac{R_1}{a_1 R_1 + a_2 R_2}\right] \leq \left[\frac{a_1 R_1 + a_2 R_2}{R_1}\right] = a_1 + (1 - a_1)\mathrm{E}\left[\frac{R_2}{R_1}\right], \tag{35}$$

implying $\mathrm{E}\left[\frac{R_2}{R_1}\right] \geq 1$.[19]

In our model, R_t^E is just one component of investors' return on wealth, which also depends on new venture returns, $R_t^{N,i}$. The return R_t^E is therefore not the sole determinant of an investor's marginal utility growth, and the CAPM does not hold with respect to R_t^E. The wedge between R_t^E and the mean-variance efficient portfolio constitutes additional risk in the denominator of the PME measure, and thus biases the outcome upwards.

4.4 The participation arc Δ and the size of the financial industry

So far we have treated the participation arc Δ as an exogenous parameter. Now we discuss how it is determined inside the model. Equation (26) determines the size of the participation arc Δ, which is a monotone function of the resources devoted to the financial industry.

To gain some intuition on the determinants of the size of the financial industry, we provide the following comparative-static results.

Lemma 1 *Δ is an increasing function of η and υ and a declining function of δ.*

Lemma 1 states that, as expected displacement increases (an increase in η), it becomes more attractive to expend resources to reduce the uncertainty associated with risky new ventures. Similarly, the lower the fraction of shares that is retained by newly arriving agents (υ), the larger the incentive to expend resources to risk-share with everyone else. Finally, a

[19] The second equality uses the fact that the pricing kernel prices both R_w and R_1, while the inequality follows from Jensen's inequality applied to the convex function $x \mapsto x^{-1}$.

higher value of δ increases the cost of the financial industry and hence the resources expended on it.

The following proposition summarizes the impact of increased displacement on the economy, taking into account the impact of endogenizing Δ.

Proposition 6 *For γ close enough to one, an increased displacement rate (a higher η) implies a larger fraction of resources devoted to the financial industry (a higher Δ) and a higher fraction of the aggregate portfolio directed towards new ventures (higher ϕ_N).*

5 Extensions and Discussion

We next sketch various extensions of the model to allow for a discussion of buyout funds and "real" assets such as real estate.

5.1 Buyout funds

We have referred to the intermediaries that facilitate risk-sharing between entrepreneurs and existing investors as "private equity" funds rather than venture capital funds. The reason is that the crucial characteristic of private equity in our model is (a) that the gains from innovation accrue (at least partially) to arriving innovator cohorts and (b) private equity funds facilitate a partial trade between existing investors and arriving entrepreneurs. By isolating these two dimensions, venture capital and buyout funds can be viewed as performing similar functions.

Specifically, to model buyout funds specifically, assume that each period a fraction of the existing firms lose their eligibility to receive new blueprints. Newly arriving entrepreneurs have the ability to purchase those firms from existing investors for a value Π per blueprint, and restore their ability to receive an allocation of blueprints over the next period, at which point the firms are re-introduced into the public market. The assumption that the new entrepreneurs have the *unique* ability to restore the ability of existing firms to receive new blueprints is equivalent to assuming that these new blueprints are effectively their property, as in the baseline model.

In summary, whether the blueprints arrive to newly created firms, or to existing firms who occasionally lose the ability to receive new blueprints, at which point they are taken private by uniquely skilled entrepreneurs, and then re-listed, is irrelevant for this model.

5.2 Real assets

So far we have studied a model where labor is the only factor of production. We now extend the model to introduce additional factors. We distinguish between two types of factors, namely those that are not tied to a specific blueprint but are useful for all productive purposes, and those that are specific to a given blueprint. An example of the first factor of production would be commercial real estate or commodities, while an example of the second factor of production would be specialized equipment required to manufacture a given intermediate good.

5.2.1. Non-displaceable factors of production

The introduction of a factor such as land is straightforward. To be as explicit as possible that land is not tied to any intermediate good, we assume that land is useful only in the production of the final good.

Land is owned by existing agents and rented out to final-good producing firms, so that aggregate output is given by

$$Y_t = F_t^{\zeta}\left(L_t^F\right)^{1-\alpha-\zeta}\left(\int_0^{A_t} x_{j,t}^{\alpha} dj\right), \tag{36}$$

where $\zeta \in (0, 1-\alpha)$ is the share of output that accrues to land. Total land is fixed and normalized to one ($F_t = 1$). Given the Cobb-Douglas structure of (36), it follows that the rental rate of land is

$$r_t^F = \zeta Y_t. \tag{37}$$

Once again, we construct an equilibrium where the price-to-rent ratio $P^F = \frac{P_t^F}{r_t^F}$ is constant, so that the return on land is given by $R_{t+1}^F = \frac{r_{t+1}^F + P_{t+1}^F}{P_t^F} = \frac{Y_{t+1}}{Y_t}\frac{1+P^F}{P^F}$. Repeating the arguments of Section 4.1, the wealth evolution of an individual investor, conditional on survival, is

$$\frac{W_{t+1}^i}{W_t^i} = \frac{1}{1-\lambda}\frac{W_{t+1}}{W_t}\left(\frac{\zeta\left(1+P^F\right)}{\alpha\left(1+P^E+P^N\right)+\zeta\left(1+P^F\right)} + \frac{\alpha\left(1+P^E\right)}{\alpha\left(1+P^E+P^N\right)+\zeta\left(1+P^F\right)}\left(\frac{1+\eta\upsilon\Gamma_{t+1}}{1+\eta\Gamma_{t+1}}\right)X_{i,t+1}\right), \tag{38}$$

for some new constants P^F, P^N, and P^E. Comparing (38) with (29), the only difference is that the wealth growth of an individual investor now gains a fraction $\frac{\zeta\left(1+P^F\right)}{\alpha\left(1+P^E+P^N\right)+\zeta\left(1+P^F\right)}$ of aggregate wealth growth. The reason is inutitive: Since land captures a constant fraction of total output, it actually benefits from higher values of Γ_{t+1}, since those are associated with higher output growth.

For sufficiently small Δ, υ, and ζ, it follows that the SDF is negatively related to Γ_{t+1}. Since the return R^E_{t+1} is declining in Γ_{t+1}, while R^F_{t+1} is increasing in Γ_{t+1}, it follows that $E\left(R^F_{t+1}\right) < E\left(R^E_{t+1}\right)$. Indeed, in addition $E\left(R^F_{t+1}\right) < 1 + r^f$, a result that, however, depends critically on the absence of neutral productivity shocks in the model.

Clearly, an increase in η will render investments in land more attractive, as a hedge to increased displacement risk.

5.2.2. Displaceable factors of production

If a factor of production was useful for the production of a fixed intermediate good (rather than being a general purpose factor that is useful in the production of the final good), its value would be vulnerable to displacement shocks. To fix ideas, assume that in addition to labor, the production of the new goods requires a location-specific capital k_i:

$$x_{i,t} = k_i^{\nu} l_{i,t}^{1-\nu},$$

where $l_{i,t}$ is the amount of labor used in the production of intermediate good i and k_i denotes an irreversible capital investment k_i that is specific to the intermediate good i. Moreover, assume that capital for the arriving vintages is produced by converting consumption goods to capital goods with one unit of investment good requiring one unit of the consumption good.

Similar to the baseline model, the financial industry allows existing investors at time Δ to invest in capital goods in an arc of length Δ without knowing which locations will be receiving blueprints in the next period. This capital is then sold to the newly arriving firms in the respective locations once production commences at time $t+1$. For simplicity (and to save notation), we assume that the entrepreneurs cannot trade blueprints behind the veil of ignorance. Also for simplicity assume that $Z_t = 1$ for all t.

We will only sketch the solution of this version of the model, since the key intuitions are no different than in the baseline model. In this version of the model aggregate output evolves according to

$$\frac{Y_{t+1}}{Y_t} = \left(\frac{A_{t+1}}{A_t}\right)^{1-(1-\nu)\alpha}$$

in a balanced growth path. The owners of capital goods extract a fraction ν of the present value of profits of the firms produced in location i, so that the return from investing in the new capital goods is given by

$$R_{i,t+1}^{N}=\frac{\nu A_{t}\left(\Pi_{t+1}+\pi_{t+1}\right)\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}}d\Gamma_{i,t+1}}{A_{t}k_{t}\left(1+\psi f\left(\Delta\right)\right)}, \tag{39}$$

which parallels closely equation (17), except that (a) existing investors obtain a fraction ν (rather than υ) of total profits, and (b) the cost of their investment is given by $A_t k_t\left(1+\psi f\left(\Delta\right)\right)$, where $A_t k_t$ is the number of investment goods and $1+\psi f\left(\Delta\right)$ is the cost per unit of capital good. Since the equilibrium of this model features a constant price-to-profits ratio in equilibrium, equation (39) can alternatively be expressed in q-theoretic fashion as

$$R_{i,t+1}^{N}=\frac{\nu\frac{\pi_{t+1}}{k_{t}}\left(1+P^{E}\right)\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}}d\Gamma_{j,t+1}}{1+\psi f\left(\Delta\right)}, \tag{40}$$

where $\nu\frac{\pi_{t+1}}{k_t}$ is the marginal product of capital at firm inception and $1+\psi f\left(\Delta\right)$ is the marginal cost of converting one unit of consumption to capital. Comparing (40) with (17) shows that ν plays a similar role to υ, in that it controls the fraction of the present value of profits that accrues to existing investors. However, unlike the baseline model, the reason why existing investors capture that fraction is that they are providing the capital goods required for production, rather than insuring the entrepreneur.

In this version of the model the equilibrium quantity that adjusts to clear markets is k_{t+1}, the quantity of capital, rather than the valuation ratio P^N. Hence, the intuitions of the baseline model that pertains to the determination of P^N carry over to the determination of k_{t+1}.

We conclude by noting that even though we drew a stark distinction between factors of production that are tied to specific intermediate goods and those that are not, in reality the distinction is more nuanced, with many factors being convertible between different uses at some cost.

5.3 Labor income and pension funds

Labor income benefits from displacement in our model since the arriving firms compete for labor services, which are in fixed supply. Indeed, total wages increase at the same rate as aggregate output. Moreover, we have assumed that workers are hand-to-mouth consumers who are not participating actively in financial markets. These assumptions are purely for simplicity and can be easily relaxed.

Suppose for instance that workers are allowed to participate in financial markets. Moreover, suppose that the production of a unit of each intermediate good takes one unit of

a^E	14.06	α	0.80
a^N	17.36	γ	9.00
a^U	1.23	Δ	0.50
b^E	0.26	υ	0.80
b^N	0.03	ρ	0.95
b^U	0.39	$\delta f(\Delta)$	0.02
$\mathrm{E}[\varepsilon]$	-0.23	$1-\lambda$	0.90
$\sigma(\varepsilon)$	0.07	$\hat{\beta}$	0.85

Table 1: Parameters used for the calibration. One model period is five years.

a labor-composite good, which is a Cobb-Douglas aggregate of labor inputs provided by different cohorts of workers

$$l_t = \prod_{s=-\infty..t} (l_{t,s})^{a_{t,s}}, \tag{41}$$

where $l_{t,s}$ is the labor input of workers born at time s and the weights $a_{t,s}$ are given by $a_{t,s} = \frac{\Delta A_s}{A_t}$. This specification implies that even though the aggregate wage bill grows at the rate of aggregate output, the fraction of wages accruing to a given cohort of workers declines over time and indeed at the same rate as the profits of existing firms.

One motivation for such a specification is skill obsolescence: As the number of blueprints expands, the skills of a given cohort of workers becomes progressively less useful.

If one were to adopt equation (41), and allowed workers access to financial markets on the same terms as firm owners, then all our conclusions would carry through without modification: With such a specification, workers' human capital would exhibit a similar exposure to displacement as the value of existing firms, making workers eager to hedge displacement risk by investing in newly arriving companies.

In particular, if one took the view that pension funds invest on behalf of workers in a way that maximizes their welfare, then their portfolios would exhibit the same patterns as the portfolios of the investors in our baseline model.

6 Calibration

In this section we examine the model's quantitative implications for asset pricing and the wealth inequality.

	Model	Data
Cons growth mean	0.021	0.018
Cons growth std dev	0.039	0.025
Equity premium, $\mathrm{E}[R^E - R^f]$	0.030	0.052
Std. deviation of public equity, $\sigma(\log(R^E))$	0.049	0.182
Std. deviation of private equity, $\sigma(\log(R_i^N))$	0.186	0.180
Cross-sectional standard deviation of $\log(R_i^N)$	0.050	0.041
Market-beta of private equity $\beta_{R^E}(R_i^N)$	0.853	1.000
Portfolio share of PE, ϕ_N	0.074	0.080
PME, $\mathrm{E}\left[\frac{R^N}{R^E}\right]$	0.084	0.100

Table 2: Model and data — various moments. Consumption growth rates and all asset returns are annualized.

6.1 Parameter choice

We choose the parameters of the model to match (a) the cross-sectional dispersion of private-equity returns for investors and (b) the share of private equity investments in a typical investor's portfolio. We then ask whether this calibrated version of the model can produce PME values that are consistent with the data.

A key aspect of our model is the heterogeneity of returns obtained by investors. Before we confront the model with the data, we remind the reader that one of our core motivating facts, laid out in Section 2, is the highly disperse distribution of returns experienced even by institutional investors in the private-equity market, in contrast to their returns in the public markets.

We next provide some details on how we choose parameters to gauge the quantitative implications of this return dispersion. To start, we choose a "period" to be five years and compare five-year returns in the model and the data (in order to mitigate the measurement errors in annual returns discussed above). For preference parameters we choose an annual discount rate of approximately 1% per year and an annual birth/death rate of 2%, which are common choices in the literature. For risk aversion we choose $\gamma = 9$, which is in the range of values that are considered in the literature. We set $\upsilon = 0.8$, to implying that entrepreneurs retain about 20% of the equity share in a typical private equity deal.[20]

[20] In a "typical" first round venture capital contract, the VC obtains about 30-40 % of equity and the founders retain 20-30%. A fraction of 20-30% accrues to Angel investors and a fraction of about 20% re-

In terms of the parameters that control output, we set α to match the share of economic rents ("pure profits") in production. In particular, the value $\alpha = 0.8$ implies a profit share of $\alpha(1-\alpha) = 16\%$. This is in line with the estimates in the literature, especially in the more recent decades. We model the neutral productivity process, Z_t, as a random walk

$$\Delta \log Z_{t+1} = \varepsilon_{t+1},$$

where ε_t is normal and independent across time. We choose the mean and standard deviation of ε to match the 5-year changes of log-output changes in the data. Note that we intentionally refer to Z_t as a neutral productivity process rather than TFP. In our model, since there is no capital and labor is fixed across time, all of the variation in output, Y_t, which is the product of Z_t and $A^{1-\alpha}$, is the (residual) part of TFP that is not driven by the arrival of new intermediate goods and is neutral in the sense that it does not cause redistribution of profits between intermediate-goods producers. By allowing for Z_t we are able to match the stochastic properties of output growth. Another practical benefit is that it allows us to match the observed beta between private equity and the public stock market. Specifically, we assume that shocks to ε_{t+1} and Γ^N_{t+1} are correlated. Since ε_{t+1} is normally distributed, while Γ^N_{t+1} is Gamma distributed, we use a Gaussian copula to model the correlation between these two random variables using a correlation of $\rho = 0.95$, which is chosen to match a private equity beta to the public-market portfolio around unity. (An alternative way to motivate this correlation is that in the data the availability of private equity funding tends to be cyclical, and so is the share of VC-backed entrepreneurs.)

The parameters α^l and β^l for $l \in \{E, N, U\}$ control the distribution of aggregate blueprints and the allocation of these blueprints across existing firms, VC-backed entrepreneurs, and non-VC-backed entrepreneurs. In terms of observable counterparts, the values of α^l and β^l for $l \in \{E, N, U\}$ are reflected in (a) the depreciation rate of existing blueprints from one period to the next, $(1 + \Gamma^E_{t+1} + \Gamma^N_{t+1} + \Gamma^U_{t+1})^{-1}$, (b) the ratio of old-to-total stock market capitalization, $(1+\Gamma^E_{t+1}) \times (1+\Gamma^E_{t+1}+\Gamma^N_{t+1}+\Gamma^U_{t+1})^{-1}$, (c) the ratio of VC-backed to non-VC-backed IPOs, $\frac{\Gamma^N_{t+1}}{\Gamma^U_{t+1}}$. Jointly with all other parameters in the model, the parameters α^l and β^l also determine (a) the variance (and higher moments) of the logarithmic excess returns of private equity, (b) the variance (and higher moments) of log output, and (c) the share of private equity in an investor's portfolio.

Instead of fully specifying the function f (and by implication f'), which captures the

mains in the option pool. Source: "https://www.entrepreneur.com/money-finance/business-dividing-equity-between-founders-and-investors/65028". By setting the entrepreneur's share to a low number, we wish to account for further dilution in future rounds of financing.

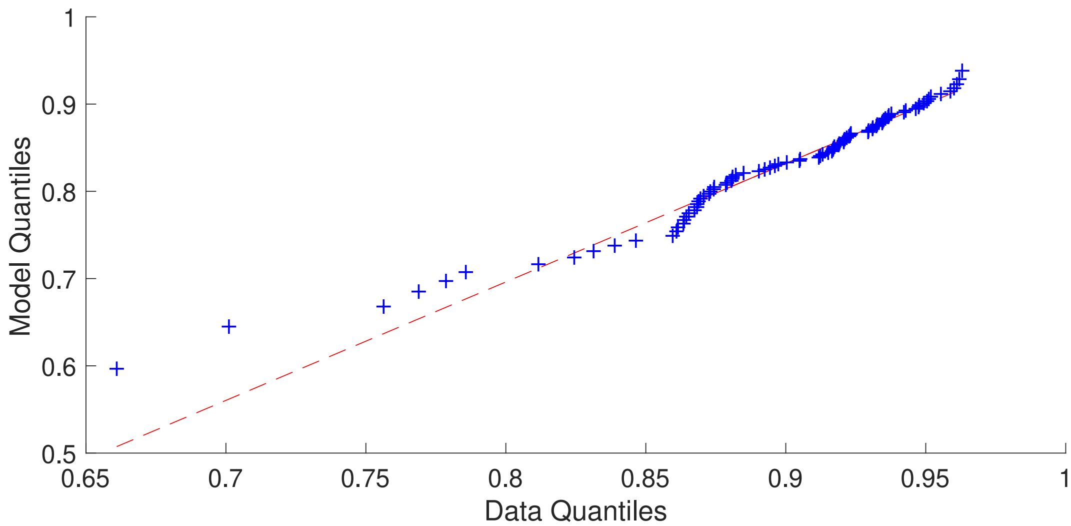

Figure 8: Quantile-quantile plot of the ratio of old-to-total stock market capitalization, $\frac{1+\Gamma^E_{t+1}}{1+\Gamma^E_{t+1}+\Gamma^N_{t+1}+\Gamma^U_{t+1}}$, in the model and the data. To compute this ratio in the data we use rolling 5-year observations.

resources absorbed by the financial industry (resp. the choice of Δ), we directly specify Δ to match the cross-sectional dispersion of private equity returns in the data and $\delta f(\Delta)$ to capture the added value of the financial industry as a share of aggregate profits. Since the financial industry for the purposes of this paper is the private-equity industry, we choose a low number. (We note that this number is actually quite inconsequential for any of the quantities of interest.)

6.2 Implications for displacement and asset prices

We next compare the model to the data starting with a discussion of the model's quantitative implications for displacement of old firms by new firms. Over a five-year period, the old-to-total stock market capitalization ratio has a mean of about 0.89 in the data (0.82 in the model) and a standard deviation of 0.074 in the data (0.087 in the model). Figure 8 provides a quantile-quantile plot of this ratio in the data and the model, which shows that the model matches the shape of the distribution of this ratio reasonably well. The model also implies a high depreciation rate for any fixed blueprint (approximately 80% over a 5-year period), which is consistent with the high depreciation rates estimated for intangible assets in the literature. (Since the majority of the new blueprints accrue to existing firms in our calibration, we can jointly match a high depreciation rate for a fixed blueprint and the slower decay of the market value of old firms).

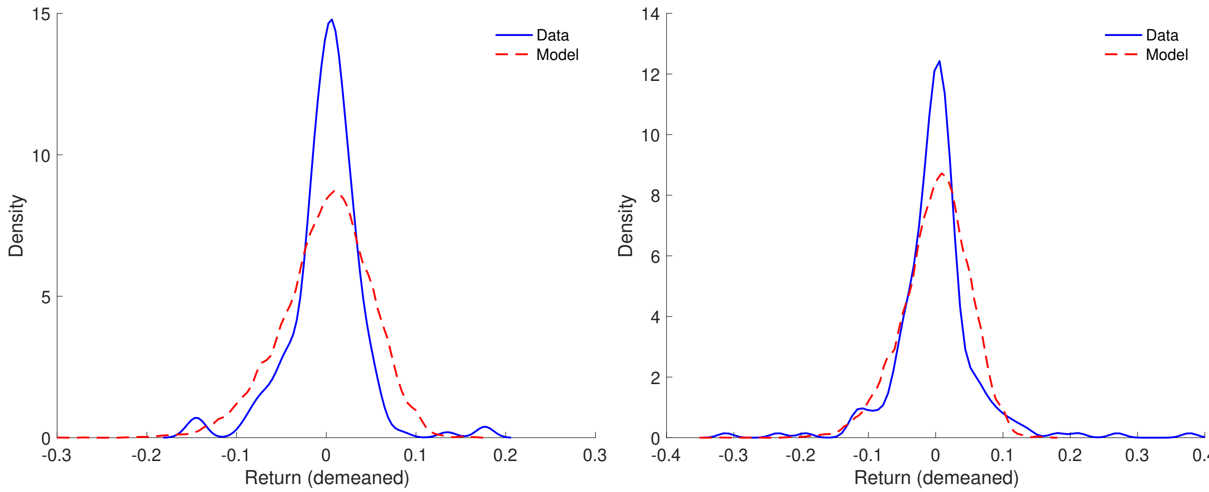

Figure 9: Dispersion of private-equity returns in the model and the data. The left plot uses the private equity returns reported in public pension plans CFAR reports, the right plot uses internal rates of returns on the private equity investments of public pension plans.

Table 2 reports more moments for the model (left column) and the data (right column). Because of the skewness that is introduced by the Gamma-distributed shocks, the model implications for the returns of existing firms resembles a "disaster-risk" model. Specifically, the model produces a sizeable (unlevered) equity premium (of the same order of magnitude as the unlevered equity premium in Barro (2008).) However, because the model abstracts from mechanisms that could make the price-dividend ratio time-varying, it cannot account for the volatility of returns of existing firms. By now there are several approaches in asset pricing to account for this volatility (heterogeneous preferences, beliefs, time-varying risk aversion, etc.), but such extensions would be just a distraction for this paper.

The model matches quite well the second moments of private equity returns, and in particular (a) the time-series volatility of private equity returns, (b) the beta of private equity to public equity returns, and (c) the cross-sectional dispersion of private equity returns across investors. Figure 9 provides a visual illustration of (c). The figure plots the cross-sectional dispersion of 5-year returns that investors obtain in private equity in the model and the data. Specifically, over every five year period we report the distribution of a pension plan's 5-year private-equity return minus the cross-sectional mean (across pension plans) over that same period. We contrast this cross-sectional dispersion in the model and the data. In the left plot we use the data provided by the CFAR reports. To ensure that this cross-sectional dispersion is not just a result of measurement error in CFAR reports, the right plot uses a different data source, namely the Prequin database, which records the deal-level internal rates of return

Portfolio	2-1	3-1	4-1	5-1	6-1	7-1	8-1	9-1	10-1
Ex. returns, data	-0.01	0.01	0.03	0.03	0.05	0.06	0.06	0.08	0.10
Ex. returns, model	0.02	0.02	0.03	0.03	0.04	0.04	0.05	0.05	0.06

Table 3: Value premium. The table reports the return to each of the top nine decile portfolios ranked by the earnings-to-price ratio (E/P) in excess of the return on the first of the decile portfolios.

(IRR) of the private equity investments made by public pension plans.[21] Irrespective of which data set we use, the figure shows that our model matches the cross-sectional dispersion of returns quite well.

Turning to the model-implied PME, we find that the model produces a PME of the same order of magnitude as in the data. This illustrates that even in the absence of any skill, PME values are above one, and the magnitude of the deviation is similar to the data. We also note that the average private-equity return exceeds the return on public equity in the model by 1.7% per annum.

Finally, we confirm that under our parameterization choice we obtain not only the return premium for new ventures, but simultaneously a value premium — a positive loading on the innovation shocks provides a hedge. To this end, we enrich the model by allowing the growth of existing firms to covary with the innovation shock $\Gamma^N + \Gamma^U$. Our construction has the same mathematical form as the well-known specification of individual endowment shocks in Constantinides and Duffie (1996). That is, we let the profit of any existing firm, denoted π_t^i, follow

$$\frac{\pi_{t+1}^i}{\pi_t^i} = \frac{\pi_{t+1}}{\pi_t}\left(1 + \eta\Gamma_{t+1}^E\right) e^{(\Gamma_{t+1}^N + \Gamma_{t+1}^U)\sigma_\zeta \zeta_{i,t+1} - \frac{1}{2}(\Gamma_{t+1}^N + \Gamma_{t+1}^U)^2 \sigma_\zeta^2}, \tag{42}$$

with $\zeta_{i,t+1}$ standard normal, independent across firms and time. When $\sigma_\zeta = 0$, all existing firms experience identical profit growth. Furthermore, for any value of the parameter σ_ζ the aggregate growth of existing firms' cashflows is the same as specified in Section 3. The value/growth characteristic of a firm is determined by a time-t signal of $\zeta_{i,t+1}$. We assume, in particular, that investors learn in which of N equal-sized bins the idiosyncratic shocks lies. For our illustration, we choose $N = 10$.

[21] We compute a five-year average of IRRs for the specific private equity investments that each pension fund made, and then subtract the cross-sectional (across pension funds) mean of this quantity over that same period.

Min Rank	350	300	250	200	150	100	50
Data	20.5	17.2	13.0	16.0	10.7	11.1	6.8
Model	11.3	11.5	11.2	12.1	12.1	13.2	14.5

Table 4: Comparison of the wealth of new entrants to old members. Every five years we report the percentage of new entrants into the Forbes 400 whose rank is between 400 and 350, 350 and 300, etc. The top row is for the data, the bottom row for the model.

We compute the average returns to portfolios consisting of all stocks in each of the ten bins, and we report the spread between each of the top nine deciles and the bottom one. For comparison, we also report the corresponding numbers from Fama and French (1992). In the data, the spread in excess returns is higher, but it is important to recall that there is no leverage in the model.

6.3 Wealth distribution

Compared to the majority of models that study the wealth distribution in the presence of heterogeneous returns, this model focuses on the role of entry and exit. In particular, the key feature of the model is that some investors (in particular, the non-VC backed entrepreneurs who turn out to be successful) experience a stratospheric accession to wealth in the first period of their life, followed by less volatile but lower average returns thereafter.

This is an empirically attractive feature of the model, as we illustrated with Figure 2. We tabulate the data plotted in that figure, as well as the model counterpart, in Table 4. Using data from the Forbes 400, we find that approximately 20% of the individuals joining the Forbes 400 list are new entrants — people who were not in any of the previous 400 lists — and who displace previous members of the Forbes 400. The remarkable feature of the data to us, which we already highlighted, is that the new entrants do not predominantly populate the lower ranks of the Forbes 400 list. Only 20.5% find themselves in the 350-400 range, while almost 20% populate the ranks 150 and above. With its very disperse distribution of wealth in the early stage of an entrepreneur's life, the model can match this remarkable phenomenon in the data, as the second row of the table shows.

Table 5 compares the stationary distribution of the model to the data. We simulate the model, normalize the log wealth of the 100th richest individual to zero, and then report the log wealth difference between the 150th to the 100th individual, the 200th to the 100th

Rank	400	350	300	250	200	150
1982	-1.2	-0.9	-0.9	-0.7	-0.5	-0.2
1992	-1.1	-1.0	-0.8	-0.7	-0.5	-0.3
2002	-1.2	-1.0	-0.8	-0.7	-0.5	-0.3
2012	-1.3	-1.1	-0.9	-0.7	-0.5	-0.3
Model	-2.4	-2.0	-1.7	-1.3	-0.9	-0.5

Table 5: Log-wealth of the richest individuals. Data is from Forbes 400. The log wealth of the 100th richest individual is normalized to zero.

individual, etc.[22] The table shows that the model leads to a more disperse wealth distribution than what we observe in the data. For instance, in the data the 250th richest individual has approximately 49.6% ($e^{-0.7}$) of the wealth of the 100th richest individual, while in the model that ratio is about 27.2% ($e^{-1.3}$). The higher dispersion of the stationary wealth distribution in the model is not too surprising, given the absence of such real-world forces as taxes and split inheritances.

7 Conclusion

This paper presents a unified risk-based asset-pricing theory of both "conventional" and "alternative" asset classes. On the production side, we make two core assumptions: (a) young firms engage in creative destruction and displace old firms, and (b) the gains to innovation are extremely skewed; only a set of measure zero of arriving firms will end up valuable. The financial sector plays a key role in the model as a device to facilitate risk sharing across arriving entrepreneurs and existing investors ("inter-cohort risk sharing") and within cohorts of existing investors ("intra-cohort risk sharing").

Investing in every single arriving new venture is infeasible. In addition, entrepreneurs are forced to retain a stake in their corporation. These two assumptions, along with the extremely skewed distribution of the gains from innovation, imply that (a) both inter- and intra-cohort risk sharing are imperfect and there is dispersion across investor returns in their private equity investments, and (b) a set of measure zero of entrepreneurs become ultra rich.

Using this model we show that: (a) It is possible to reconcile some seemingly contradictory

[22]We choose the 100th individual as the base for comparison, because the log wealth of the richest, 2nd richest, etc. individuals is estimated with error even if we simulate the model for a very long time.

patterns in the returns to private and public equities. In particular, the value premium, which requires that public-equity growth options command a low risk premium, is consistent with the seemingly high risk premium commanded by growth-option intensive venture capital investments. (b) The "public market equivalent" approach to risk-adjusting the returns of private equity is biased upwards and therefore has an expected value higher than one. This result holds even though all expected returns in our model reflect exclusively compensation for risk. Thus, PME values above unity are not necessarily an indication of outperformance. (c) The wealth dispersion among the ultra rich forms at an early stage. The cohort of new entrants into the distribution of the ultra rich have a wealth dispersion not too dissimilar with the existing rich. This suggests that entry and displacement are important forces to explain the dynamics of the wealth distribution of the ultra rich; the wealth growth of the existing rich plays a secondary role. (d) If displacement risk rises (be it because of increased displacement activity or because of higher concentration of the gains from innovation) then the private equity industry grows, precautionary savings increase (making the interest rate decline for plausible parameters), and investments in land and natural resources become attractive as displacement-risk hedges.

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A Proofs

Proof of Proposition 1. A first-order condition for portfolio choice for an investor with logarithmic preferences is

$$\mathrm{E}\left[\left(\phi_B\left(1+r^f\right)+\phi_E R_t^E+\phi_N R_t^N\right)^{-\gamma}\left(R_t^E-R_t^N\right)\right]=0. \tag{43}$$

Using the definitions of ϕ_E and ϕ_N and imposing $\phi_B = 0$ and market clearing in the stock markets implies $\phi_E = \frac{P^E}{P^E+\hat{P}^N}$ and $\phi_N = 1-\phi_E$. Accordingly, using (14) and (17),

$$\phi_B\left(1+r^f\right)+\phi_E R_t^E+\phi_N R_t^N=\frac{P^E}{P^E+\hat{P}^N}R_t^E\left(1+H_t\right). \tag{44}$$

Using (44) and (17) inside (43) and noting that $\frac{P^E}{\hat{P}^N}=\frac{\phi_E}{1-\phi_E}$ leads to (20).

Having determined ϕ_E, it is straightforward to determine P^E and P^N. To start, we note that with unitary IES $c = 1-\beta(1-\lambda)$. Aggregating (10) across agents and imposing asset market clearing implies

$$W_t = A_t\pi_t\left(P^E+P^N+1\right). \tag{45}$$

Given goods market clearing $\frac{C_t}{A_t\pi_t}=(1-\delta f(\Delta))$, we obtain

$$1+P^E+P^N=\frac{1-\delta f(\Delta)}{1-\beta(1-\lambda)}. \tag{46}$$

Combining (46) with $\phi_E=\frac{P^E}{P^E+\hat{P}^N}$ results in (23)–(24).

The first-order condition for the excess return $R_t^E-\left(1+r^f\right)$ yields (25).

Finally, equations (26) and (27) follow from maximizing utility over the right-hand side of (44), making use of the envelope theorem and recalling $\hat{P}^N = P^N + \delta f(\Delta)$. ■

Proof of Proposition 2. Equation (28) follows from (45) and (5). To arrive at equation (29), we use (11), (44), (14), and the goods market clearing condition $c\frac{W_t}{A_t\pi_t}+\delta f(\Delta)=1$, which implies

$$1-c=\frac{P^E+P^N+\delta f(\Delta)}{1+P^E+P^N}.$$

■

Proof of Proposition 3. When $\upsilon = 1$ and $\Delta = 1$, $\frac{M_{t+1}}{M_t}\propto\left(1+\eta\Gamma_{t+1}^E+\eta\Gamma_{t+1}^N\right)^{\gamma(\alpha-1)}$, which is decreasing in Γ_{t+1}^N.

To prove the second claim, note first that, for $\upsilon = 0$,

$$\frac{M_{t+1}}{M_t} \propto \left(R^E_{t+1}\right)^{-\gamma} \propto \left(1+\eta\Gamma^E_{t+1}\right)^{-\gamma}\left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}\right)^{\alpha\gamma}, \tag{47}$$

so that $\frac{M_{t+1}}{M_t}$ is obviously negatively correlated with R^E_{t+1} and positively correlated with

$$\lim_{\upsilon\to 0}\frac{R^N_{t+1}}{\upsilon} = \left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}\right)^{-\alpha}\Gamma^N_{t+1}\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{\frac{i+\Delta}{2}} dL_{j,t+1}. \tag{48}$$

The last statement is due to the fact the right-hand sides of both (47) and (48) are decreasing in Γ^E_{t+1} and increasing in Γ^N_{t+1}. We invoke continuity in υ to establish the desired covariance signs for $\upsilon > 0$ small enough.

Finally, for the case of small Δ, we are going to use the following result, which we formalize as a lemma, and prove after the end of the proof to the proposition.

Lemma 2 *The random variable $\frac{1}{\Delta}\int_0^\Delta dL_{j,t}$ tends to zero in probability as Δ tends to zero. That is, for every $\varepsilon > 0$*

$$\lim_{\Delta\to 0} Prob\left(\frac{1}{\Delta}\int_0^\Delta dL_{j,t} < \epsilon\right) = 1. \tag{49}$$

Note now that, for any $\upsilon > 0$ and $\Delta > 0$,

$$\frac{M^i_{t+1}}{M^i_t} \propto \left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}\right)^{\alpha\gamma}\left(1+\eta\Gamma^E_{t+1}+\frac{\eta\upsilon}{\Delta}\Gamma^N_{t+1}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}\right)^{-\gamma}. \tag{50}$$

For $\frac{\upsilon}{\Delta}\Gamma^N_{t+1}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1} < 1$, which can be ensured to obtain with probability arbitrarily close to 1 by choosing Δ low enough,[23] the last expression decreases in Γ^E_{t+1}, while R^E_{t+1} continues to increase in Γ^E_{t+1}. It only remains to note that the contribution to the covariance of the complementary event tends to 0 as its probability does.

For the covariance with R^N_{t+1}, we note that the right-hand side of (50) increases in Γ^N_{t+1} on the event

$$\alpha\left(1+\eta\Gamma^E_{t+1}+\frac{\eta\upsilon}{\Delta}\Gamma^N_{t+1}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}\right) > \left(1+\eta\Gamma^E_{t+1}+\eta\Gamma^N_{t+1}\right)\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}. \tag{51}$$

The probability of this event goes to 1 as $\Delta \to 0$; the only states excluded are those in which Γ^N_{t+1} is larger than a bound that can be increased arbitrarily and those in which $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1} > \varepsilon$ arbitrarily chosen.

[23] $\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$ and $\int_0^\Delta dL_{j,t+1}$ have the same distribution and are both independent of Γ^N_{t+1} and Γ^E_{t+1}.

Furthermore, (50) decreases in Γ^E_{t+1} for small Δ with arbitrarily high probability, as remarked above. Consequently, computing the covariance on the event on which $\frac{M_{t+1}}{M_t}$ displays the desired dependence on the two shocks yields in a strictly positive result for Δ small enough, while the contribution of the complement becomes arbitrarily small. ■

Proof of Lemma 2. The distribution of $\int_0^\Delta dL_{j,t}$ is beta with parameters $a^N\Delta$ and $a^N(1-\Delta)$. We wish to estimate

$$Prob\left(\int_0^\Delta dL_{j,t} < \varepsilon\Delta\right) = \frac{\Gamma(a^N)}{\Gamma(a^N\Delta)\Gamma(a^N(1-\Delta))}\int_0^{\varepsilon\Delta} x^{a^N\Delta-1}(1-x)^{a^N(1-\Delta)-1}dx, \tag{52}$$

which has the same limit as $\Delta \to 0$ as

$$\lim_{\Delta\to 0}\frac{1}{\Gamma(a^N\Delta)}\int_0^{\varepsilon\Delta} x^{a^N\Delta-1}dx = \lim_{y\to 0}\frac{1}{\Gamma(y)}\int_0^{\frac{y\varepsilon}{a^N}} x^{y-1}dx = \lim_{y\to 0}\frac{(\varepsilon/a^N)^y y^y}{y\Gamma(y)} = 1 \tag{53}$$

since both the numerator and the denominator tend to 1. ■

Proof of Proposition 4. We take $\eta = 1$ for simplicity. Let us first note that, given that $\mathrm{E}[M_i R^E] = \mathrm{E}[M_i R^{N,i}]$,

$$\frac{\mathrm{E}[R^{N,i}]}{\mathrm{E}[R^E]} > 1 \quad \Leftrightarrow \quad \frac{\mathrm{E}[M_i R^{N,i}]}{\mathrm{E}[M_i R^E]} < \frac{\mathrm{E}[R^{N,i}]}{\mathrm{E}[R^E]}. \tag{54}$$

This fact allows us to replace each of $R^{N,i}$ and R^E with an arbitrary (scalar) multiple of itself.

$$R^E \propto \xi^E \equiv Z(1+\Gamma^E+\Gamma^N+\Gamma^U)^{-\alpha}(1+\Gamma^E) \tag{55}$$

$$R^{N,i} \propto \xi^{N,i} \equiv Z(1+\Gamma^E+\Gamma^N+\Gamma^U)^{-\alpha}\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j \tag{56}$$

$$M^i \propto \zeta^i \equiv Z(1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha\gamma}\left(1+\Gamma^E+\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j\right)^{-\gamma}, \tag{57}$$

so and therefore replace (54) with

$$\frac{\mathrm{E}[R^{N,i}]}{\mathrm{E}[R^E]} > 1 \quad \Leftrightarrow \quad \frac{\mathrm{E}[\zeta^i\xi^{N,i}]}{\mathrm{E}[\zeta^i\xi^E]} < \frac{\mathrm{E}[\xi^{N,i}]}{\mathrm{E}[\xi^E]}. \tag{58}$$

We note also that we can set $Z = 1$ from now on without affecting any of the arguments.

(i) For this part, in the second inequality in (54) we first cancel the multiplicative factor υ in $\xi^{N,i}$, then take $\upsilon = 0$, which eliminates the last term inside the second parenthesis on

the right-hand side of (57). In words, ζ^i does not depend on L. As a consequence, the term $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j$ is independent of all other random variables, so that the expectations in the two numerators factor and L does not matter.

We would like to show that

$$\frac{\mathrm{E}\left[(1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)}(1+\Gamma^E)^{-\gamma}\Gamma^N\right]}{\mathrm{E}\left[(1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)}(1+\Gamma^E)^{1-\gamma}\right]} > \frac{\mathrm{E}\left[(1+\Gamma^E+\Gamma^N+\Gamma^U)^{-\alpha}\Gamma^N\right]}{\mathrm{E}\left[(1+\Gamma^E+\Gamma^N+\Gamma^U)^{-\alpha}(1+\Gamma^E)\right]}, \tag{59}$$

which is the same as

$$\frac{\mathrm{E}\left[(\xi^E)^{1-\gamma}\frac{\Gamma^N}{1+\Gamma^E}\right]}{\mathrm{E}\left[(\xi^E)^{1-\gamma}\right]} > \frac{\mathrm{E}\left[\xi^E\frac{\Gamma^N}{1+\Gamma^E}\right]}{\mathrm{E}\left[\xi^E\right]}. \tag{60}$$

We note that R^E increases with Γ^E and decreases with Γ^N, precisely the opposite pattern from $\frac{\Gamma^N}{1+\Gamma^E}$. We use the following lemma.

Lemma 3 *Let X_i, $i = 1,\dots,n$, $n \geq 1$, be independent (one-dimensional) random variables and functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ increasing in all of their arguments. Then*

$$Cov(f(X_1,\dots,X_n), g(X_1,\dots,X_n)) \geq 0. \tag{61}$$

In our context, we set $X_1 \equiv \Gamma^U$, $X_2 \equiv (1+\Gamma^E)^{-1}$, and $X_3 \equiv \Gamma^U$, and treat $-\xi^E$ as an increasing function in each X_i, as is $\Gamma^N/(1+\Gamma^E)$. We then apply the same logic to $(\xi^E)^{1-\gamma}$ as f. Consequently, we have

$$\mathrm{Cov}\left(\xi^E, \frac{\Gamma^N}{1+\Gamma^E}\right) < 0 < \mathrm{Cov}\left((\xi^E)^{1-\gamma}, \frac{\Gamma^N}{1+\Gamma^E}\right). \tag{62}$$

Inequality (60) now follows using the definition of covariance.

(ii) For this part we rely on the fact that $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j$ tends to 0 in probability. The main implication is that $\mathrm{E}[\zeta^i\xi^{N,i}]$ tends to 0, where $\zeta^i\xi^{N,i}$ is given by

$$\begin{aligned}(1+\Gamma^E+&\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)}\left(1+\Gamma^E+\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j\right)^{-\gamma}\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j \\ &\geq (1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)}\left(1+\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j\right)^{-1}\Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j.\end{aligned} \tag{63}$$

To show this result, define the random variables X and Y as

$$X = (1+\Gamma^E+\Gamma^N+\Gamma^U)^{\alpha(\gamma-1)} \tag{64}$$

$$Y = \Gamma^N\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_j \tag{65}$$

and note that Y tends to zero in probability as Δ goes to 0.

For an arbitrary ε, choos Δ such that $Y < \varepsilon$ with probability $1-\varepsilon$. We have

$$\begin{aligned}
\mathrm{E}\left[X\frac{Y}{1+Y}\right] &= \mathrm{E}\left[X1_{X<\varepsilon^{-\frac{1}{2}}}\frac{Y}{1+Y}1_{Y<\varepsilon}\right] + \mathrm{E}\left[X1_{X<\varepsilon^{-\frac{1}{2}}}\frac{Y}{1+Y}1_{Y\geq\varepsilon}\right] + \\
&\quad \mathrm{E}\left[X1_{X\geq\varepsilon^{-\frac{1}{2}}}\frac{Y}{1+Y}\right] \\
&\leq \mathrm{E}\left[\varepsilon^{-\frac{1}{2}}1_{X<\varepsilon^{-\frac{1}{2}}}\varepsilon 1_{Y<\varepsilon}\right] + \mathrm{E}\left[\varepsilon^{-\frac{1}{2}}1_{X<\varepsilon^{-\frac{1}{2}}}1_{Y\geq\varepsilon}\right] + \mathrm{E}\left[X1_{X\geq\varepsilon^{-\frac{1}{2}}}\right] \\
&\leq \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} + \mathrm{E}\left[X1_{X\geq\varepsilon^{-\frac{1}{2}}}\right], \qquad (66)
\end{aligned}$$

where the very last term tends to zero as $\varepsilon \to 0$ since X has finite mean.

On the other hand, $\mathrm{E}[\zeta^i\xi^E]$ and $\mathrm{E}[\xi^{N,i}]$ are bounded below away from 0, implying that $\mathrm{E}[R^{N,i}] > \mathrm{E}[R^E]$. (We note that the proof also implies that, as $\Delta \to 0$, $\mathrm{E}[R^{N,i}] \to \infty$, a fact we already invoked in stating that entrepreneurs have a value of zero for shares in their own firm.) ■

Proof of Lemma 3. Start by noting

$$E[fg] = \mathrm{E}[\mathrm{E}[fg|X_n]] \geq \mathrm{E}[\mathrm{E}[f|X_n]\mathrm{E}[g|X_n]], \qquad (67)$$

where we are making use of the "induction hypothesis" that the result holds for $n-1$ variables (for each realization of X_n). It also holds that $\mathrm{E}[f|X_n]$ and $\mathrm{E}[g|X_n]$ are increasing functions of X_n, however, so that we can apply the result with $n = 1$ (the "classical" result) to infer

$$\mathrm{E}[\mathrm{E}[f|X_n]\mathrm{E}[g|X_n]] \geq \mathrm{E}[\mathrm{E}[f|X_n]]\mathrm{E}[\mathrm{E}[g|X_n]] = E[f]\mathrm{E}[g]. \qquad (68)$$

We now combine the two inequalities to obtain the desired conclusion. ■

Proof of Proposition 5. The proof for the log case ($\gamma = 1$) is contained in the text. As long as $R^{N,i}$ and R^E are not fully correlated, the expectation $\mathrm{E}\left[\frac{R^{N,i}}{R^E}\right]$ is strictly larger than 1 when $\gamma = 1$, and therefore by continuity also for γ close enough to 1. ■

Proof of Lemma 1. It suffices to show that

$$\frac{\partial^2 \mathrm{E}\left[\log\left(1+\eta\Gamma_t^E+\eta\Gamma_t^U+\frac{\eta v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N\right)\right]}{\partial\Delta\partial\eta} > 0. \qquad (69)$$

Differentiating first with respect to η gives

$$\frac{\partial \mathrm{E}\left[\log\left(1+\eta\Gamma_t^E+\eta\Gamma_t^U+\frac{\eta v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N\right)\right]}{\partial\eta} = \mathrm{E}\left[\frac{\Gamma_t^E+\Gamma_t^E+\frac{v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N}{1+\eta\Gamma_t^E+\eta\Gamma_t^U+\frac{\eta v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N}\right].$$

The function $x \mapsto \frac{x}{1+\eta x}$ is concave. Further, since $\Gamma_t^E + \Gamma_t^U$ and $\frac{v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N$ are independent and $\frac{v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N$ increases in Δ in the sense of second-order stochastic dominance, so does $\Gamma_t^E + \Gamma_t^U + \frac{v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N$. Inequality (69) follows. ■

Proof of Proposition 6. The only result remaining to prove is the dependence of $\phi_N = 1 - \phi_E$ on η and Δ. In the logarithmic utility case,

$$\phi_E = \mathrm{E}\left[\left(1 + \left(1+\eta\Gamma_t^E\right)^{-1}\eta\frac{v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N\right)^{-1}\right] \tag{70}$$

$$= \mathrm{E}\left[\left[\left(1 + \left(1+\eta\Gamma_t^E\right)^{-1}\eta\frac{v}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t}^N\right)^{-1}\middle|\Gamma_t^E\right]\right], \tag{71}$$

and the quantity inside the square brackets in (70) decreases in η for all realizations of the (positive) random variables Γ_t^E and $d\Gamma_{j,t}^N$. An increase in Δ, just as in the proof of Lemma 1, results in a second-order stochastically dominant random variables in (70), so that, for every realization of Γ_t^E, the conditional expectation in (71) decreases. We conclude that ϕ_N increases with η. ■