

Dynamic Portfolio Choice with Frictions: Appendix

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March 26, 2014

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A Proofs

Proof of Proposition 1. This proposition is a special case of Proposition 2, but its simplicity allows for the relaxation the constraints placed on other parameters to obtain a well-behaved problem. Specifically, it is immediate that the objective (5) is strictly concave in $\{\tau_s\}_s$. To prove optimality with an infinite horizon, we impose a transversality condition on any admissible strategy, namely that

$$\lim_{T \rightarrow \infty} E_t[e^{-\rho(T-t)} x_T] = 0. \quad (\text{A.1})$$

We also impose appropriate conditions ensuring that f is stationary.

The Hamilton-Jacoby-Bellman (HJB) equation is

$$\rho V = \sup_{\tau} \left\{ x^\top B f - \frac{\gamma}{2} x^\top \Sigma x - \frac{1}{2} \tau^\top \Lambda \tau + \frac{\partial V}{\partial x} \tau + \frac{\partial V}{\partial f} \mu_f + \frac{1}{2} \text{tr} \left(\Omega \frac{\partial^2 V}{\partial f \partial f^\top} \right) \right\}. \quad (\text{A.2})$$

Maximizing this expression with respect to the trading intensity results in

$$\tau = \Lambda^{-1} \frac{\partial V}{\partial x}.$$

Given the conjectured form (6) of the value function, the optimal choice τ equals

$$\tau_t = -\Lambda^{-1} A_{xx} x_t + \Lambda^{-1} A_x(f_t).$$

Once this expression is inserted in the HJB equation, it results in the following equations defining the value-function coefficients (using the symmetry of A_{xx}):

$$-\rho A_{xx} = A_{xx} \Lambda^{-1} A_{xx} - \gamma \Sigma \quad (\text{A.3})$$

$$\rho A_x(f) = -A_{xx} \Lambda^{-1} A_x(f) + \mathcal{D} A_x(f) + B f \quad (\text{A.4})$$

$$\rho A(f) = A_{xf}^\top \Lambda^{-1} A_{xf} + \mathcal{D} A_{ff}. \quad (\text{A.5})$$

Pre- and post-multiplying (A.3) by $\Lambda^{-\frac{1}{2}}$, we obtain

$$-\rho Z = Z^2 + \frac{\rho^2}{4} I - U, \quad (\text{A.6})$$

that is,

$$\left(Z + \frac{\rho}{2} I \right)^2 = U, \quad (\text{A.7})$$

where

$$Z = \Lambda^{-\frac{1}{2}} A_{xx} \Lambda^{-\frac{1}{2}} \quad (\text{A.8})$$

$$U = \gamma \Lambda^{-\frac{1}{2}} \Sigma \Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4} I. \quad (\text{A.9})$$

This leads to

$$Z = -\frac{\rho}{2} I + U^{\frac{1}{2}} \geq 0, \quad (\text{A.10})$$

implying that

$$A_{xx} = -\frac{\rho}{2} \Lambda + \Lambda^{\frac{1}{2}} \left(\gamma \Lambda^{-\frac{1}{2}} \Sigma \Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4} \right)^{\frac{1}{2}} \Lambda^{\frac{1}{2}}. \quad (\text{A.11})$$

The value of A_x follows as a solution to the ODE (A.4). Note that, using the Feynman-Kac formula, A_x can be written as

$$A_x(f) = \mathbb{E} \left[\int_0^\infty e^{-(\rho + A_{xx} \Lambda^{-1})t} B f_t dt \mid f_0 = f \right] \quad (\text{A.12})$$

$$= \int_0^\infty e^{-(\gamma \Sigma) A_{xx}^{-1} t} (\gamma \Sigma) \mathbb{E} [M_t \mid f_0 = f] dt \quad (\text{A.13})$$

$$= \int_0^\infty (\gamma \Sigma) e^{-A_{xx}^{-1} (\gamma \Sigma) t} \mathbb{E} [M_t \mid f_0 = f] dt, \quad (\text{A.14})$$

where the second equality holds because of (A.3).

If $\mu_f(f) = -\phi f$, then $A_x(f) = A_{xf} f$ and $A(f) = f^\top A_{ff} f$, and (A.4)–(A.5) become

$$\rho A_{xf} = -A_{xx} \Lambda^{-1} A_{xf} - A_{xf} \Phi + B \quad (\text{A.15})$$

$$\rho A_{ff} = A_{xf}^\top \Lambda^{-1} A_{xf} - 2A_{ff} \Phi. \quad (\text{A.16})$$

The solution for A_{xf} follows from Equation (A.15), using the general rule that $\text{vec}(XYZ) = (Z^\top \otimes X) \text{vec}(Y)$:

$$\text{vec}(A_{xf}) = (\rho I + \Phi^\top \otimes I_K + I_S \otimes (A_{xx} \Lambda^{-1}))^{-1} \text{vec}(B).$$

If $\Lambda = \lambda \Sigma$, then $A_{xx} = a \Sigma$ with

$$-\rho a = a^2 \frac{1}{\lambda} - \gamma, \quad (\text{A.17})$$

with solution

$$a = -\frac{\rho}{2} \lambda + \sqrt{\gamma \lambda + \frac{\rho^2}{4} \lambda^2}. \quad (\text{A.18})$$

In this case, (A.4) yields

$$\begin{aligned} A_{xf} &= B \left(\rho I + \frac{a}{\lambda} I + \Phi \right)^{-1} \\ &= B \left(\frac{\gamma}{a} I + \Phi \right)^{-1}, \end{aligned}$$

where the last equality uses (A.17).

Then, we have

$$\tau_t = \frac{a}{\lambda} \left[\Sigma^{-1} B (a\Phi + \gamma I)^{-1} f_t - x_t \right]. \quad (\text{A.19})$$

It is clear from (A.18) that $\frac{a}{\lambda}$ decreases in λ and increases in γ . ■

Proof of Lemma 1. The trader's utility dependence on τ is given by

$$\int_0^\infty e^{-\rho s} \left(x_s^\top (Bf_s - (r+R)D_s + C\tau_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s - \frac{1}{2} \tau_s^\top \Lambda \tau_s \right) ds, \quad (\text{A.20})$$

which is clearly concave if

$$\int_0^\infty e^{-\rho s} \left(x_s^\top (-(r+R)D_s + C\tau_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \quad (\text{A.21})$$

is. To see the latter fact, we start by evaluating¹

$$\begin{aligned} - \int_0^\infty e^{-\rho s} x_s^\top (r+R)D_s &= - \int_0^\infty e^{-\rho s} \left(\int_0^s \tau_t^\top dt (r+R) \int_0^s e^{-R(s-u)} C\tau_u du \right) ds \\ &= - \frac{r+R}{\rho+R} \int_0^\infty \int_0^\infty \tau_t^\top e^{Ru-(R+\rho)(t\vee u)} C\tau_u du dt \quad (\text{A.22}) \end{aligned}$$

and break down (A.22) in two terms depending on whether $u < t$ or vice-versa. On the set $u \geq t$ we obtain the integral

$$\int_0^\infty \int_0^u \tau_t^\top e^{-\rho u} C\tau_u dt du = \int_0^\infty x_u^\top e^{-\rho u} C\tau_u du \quad (\text{A.23})$$

$$= [x_u^\top e^{-\rho u} Cx_u]_0^\infty - \int_0^\infty x_u^\top e^{-\rho u} C (\tau_u - \rho x_u) du \quad (\text{A.24})$$

$$= \frac{1}{2} \int_0^\infty \rho e^{-\rho u} x_u^\top Cx_u du, \quad (\text{A.25})$$

¹We take $x_0 = D_0 = 0$ without loss of generality, since these values do not affect the concavity of the objective.

where the second equality follows by integration by parts and the third by solving the equation implicit in the second.

The integral on the set $u < t$ is similarly shown to be negative definite. To ensure the concavity of (A.21), it therefore suffices that

$$\frac{1}{2} \left(1 - \frac{r+R}{\rho+R}\right) \int_0^\infty \rho e^{-\rho u} x_u^\top C x_u du - \frac{\gamma}{2} \int_0^\infty \rho e^{-\rho u} x_u^\top \Sigma x_u du \quad (\text{A.26})$$

is concave, or

$$\gamma > \frac{\rho-r}{\rho+R} \left\| \Sigma^{-\frac{1}{2}} C \Sigma^{-\frac{1}{2}} \right\|_2. \quad (\text{A.27})$$

■

Proof of Proposition 2. In this case, the conjectured value function is

$$V = -\frac{1}{2} x^\top A_{xx} x + x^\top A_{xD} D + \frac{1}{2} D^\top A_{DD} D + x^\top A_x(f) + D^\top A_D(f) + A(f). \quad (\text{A.28})$$

Given the HJB equation

$$\rho V = \sup_\tau \left\{ -\frac{\gamma}{2} x^\top \Sigma x + x^\top (Bf + C\tau) - \frac{1}{2} \tau^\top \Lambda \tau + \tau^\top V_x^\top + (C\tau - RD)^\top V_D^\top + x^\top \mathcal{D}A_x(f) + D^\top \mathcal{D}A_D(f) + \mathcal{D}A(f) \right\}, \quad (\text{A.29})$$

the optimal trade follows as

$$\begin{aligned} \tau &= \Lambda^{-1} (A_x(f) + C^\top A_D(f) + (A_{xD} + C^\top A_{DD}) D - (A_{xx} + C^\top + C^\top A_{Dx}) x) \\ &\equiv \bar{M}^{rate} (\bar{M}_f^{aim}(f) + \bar{M}_D^{aim} D - x). \end{aligned} \quad (\text{A.30})$$

Plugging (A.30) in (A.29) and then proceeding as in part (i), we obtain

$$\begin{aligned} [A_x(f)^\top \quad A_D(f)^\top]^\top &= \mathbb{E} \left[\int_0^\infty e^{-N_1 t} [B^\top \quad 0]^\top f_t dt | f_0 = f \right] \\ &= \int_0^\infty e^{-N_1 t} [\gamma \Sigma \quad 0]^\top \mathbb{E} [M_t | f_0 = f] dt, \end{aligned} \quad (\text{A.31})$$

where

$$N_1 = \rho + \begin{bmatrix} A_{xx} - A_{xD}C - C \\ -A_{xD}^\top - A_{DD}C \end{bmatrix} \Lambda^{-1} [I \quad C^\top] + \begin{bmatrix} 0 & 0 \\ 0 & R^\top \end{bmatrix}. \quad (\text{A.32})$$

Equation (12) follows immediately with

$$N_2 = (A_{xx} + C^\top + C^\top A_{Dx})^{-1} [I \quad C^\top] \quad (\text{A.33})$$

$$N_3 = [\gamma \Sigma \quad 0]^\top. \quad (\text{A.34})$$

To write explicitly the Riccati equations for the constant value-function coefficients and the ODEs for the ones that depend (non-linearly) on f , we match the coefficients in the HJB equation evaluated at the optimal τ .

The constant coefficient matrices solve the system

$$\begin{aligned} -\rho A_{xx} &= -\gamma \Sigma + Q_x^\top \Lambda^{-1} Q_x \\ \rho A_{xD} &= Q_x^\top \Lambda^{-1} Q_D - A_{xD} R \\ \rho A_{DD} &= Q_D^\top \Lambda^{-1} Q_D - A_{DD} R - R^\top A_{DD}, \end{aligned} \quad (\text{A.35})$$

while the ODE system is

$$\begin{aligned} \rho A_x(f) &= Bf - Q_x^\top \Lambda^{-1} Q_f + \mathcal{D}A_x(f) \\ \rho A_D(f) &= Q_D^\top \Lambda^{-1} Q_f - R^\top A_{Df} + \mathcal{D}A_D(f) \\ \rho A(f) &= \frac{1}{2} Q_f^\top \Lambda^{-1} Q_f + \mathcal{D}A(f). \end{aligned} \quad (\text{A.36})$$

Here we used the notation

$$\begin{aligned} Q_x &= -A_{xx} + C^\top A_{xD}^\top + C^\top \\ Q_D &= A_{xD} + C^\top A_{DD} \\ Q_f &= A_x(f) + CA_D(f). \end{aligned} \quad (\text{A.37})$$

We note that the equations above have to be solved simultaneously for A_{xx} , A_{xD} , and A_{DD} ; there is no closed-form solution in general. The complication is due to the fact that current trading affects the persistent price component D (that is, $C \neq 0$). Furthermore, the ODEs for A_x and A_D are coupled, but the solution can be written reasonably simply as (A.31) above. ■

Proof of Proposition 3.

(iii) Let's start with the complete problem:

$$\begin{aligned} V(x_t, D_t, f_t) &= E_t \int_t^\infty e^{-\rho(s-t)} \left(x_s^\top (Bf_s - (r + R)D_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \\ &\quad + E_t \int_t^\infty e^{-\rho(s-t)} x_{s-}^\top C dx_s + \frac{1}{2} E_t \int_t^\infty e^{-\rho(s-t)} d[x_s, Cx_s]. \end{aligned} \quad (\text{A.38})$$

As is customary with such problems, we write and solve the HJB equation, then use the fact that it is satisfied to provide a so-called verification argument for the

proposed optimal control and value function. We also use the conjecture (22) and introduce the notation $\hat{V}(D, f) = V(0, D, f)$, so that

$$V(x, D, f) = \hat{V}(D - Cx, f) - \frac{1}{2}x^\top Cx. \quad (\text{A.39})$$

Note that

$$d(D_s - Cx_s) = -RD_s ds, \quad (\text{A.40})$$

so that $D^0 \equiv D - Cx$ is a continuous and finite-variation process.

The HJB equation is

$$\begin{aligned} 0 = \sup_{\Delta x, \mu, \sigma} & \left\{ x^\top (Bf - (r^f + R)D) - \frac{\gamma}{2}x^\top \Sigma x + x_-^\top C \frac{1}{dt} \mathbb{E}_t [dx_t] + \frac{1}{2} \frac{1}{dt} \mathbb{E}_t d[x_t, Cx_t] \right. \\ & - \rho \hat{V} + \frac{1}{2} \rho x^\top Cx + \hat{V}_D (-RD) + \hat{V}_f \mu_f \\ & - x_-^\top C \frac{1}{dt} \mathbb{E}_t [dx_t] - \frac{1}{2} \frac{1}{dt} \mathbb{E}_t d[x_t, Cx_t] + \frac{1}{dt} \mathbb{E}_t d[f_t, V_{ff} f_t] \\ & \left. + \frac{1}{\Delta t} \mathbb{E}_t \left[x_-^\top C \Delta x + \frac{1}{2} \Delta x^\top C \Delta x + \hat{V}(D^0, f_- + \Delta f) - \hat{V}(D^0, f_-) \right] \right\}. \end{aligned} \quad (\text{A.41})$$

Here, we suppressed the notational dependence on time and also wrote x_- for x_{t-} and similarly D_- and f_- .

We conjecture a quadratic form for the value function \hat{V} :

$$\hat{V} = \frac{1}{2} D^{0\top} A_{DD} D^0 + D^{0\top} A_D(f) + A(f), \quad (\text{A.42})$$

which leads to the simplification

$$\begin{aligned} 0 = \sup_x & \left\{ -\rho \hat{V} + \frac{\rho}{2} x^\top Cx + x^\top Bf - x^\top (r^f + R)(D^0 + Cx) - \frac{\gamma}{2} x^\top \Sigma x \right. \\ & \left. - \hat{V}_D R(D^0 + Cx) + D^{0\top} \mathcal{D}A_D(f) + \mathcal{D}A(f) \right\}. \end{aligned} \quad (\text{A.43})$$

We remark on the fact that (A.43) has the standard continuous-time form. The first two terms in (A.43) equal the value function decay rate $-\rho V(\hat{x}, \hat{D}, f)$, while the remaining terms represent the flow benefit from taking position x for the next infinitesimal time period: the expected excess return, the distortion decay summed with the opportunity cost of funds (the risk-free rate), from which the position x will suffer over dt , the risk cost, and the change over time in V induced by the decay of D and of f , as well as the convexity and jump adjustments for f . Note that, in order for the problem to be well defined, it is necessary that $\rho C < 2(r^f + R)C + \gamma \Sigma$ — otherwise, the agent gains too much from pushing the prices up currently relative to

the perceived cost of the risk and the decay in the distortion.

In order to write down the solution, let

$$J = \frac{1}{2}(J_0 + J_0^\top) \quad (\text{A.44})$$

$$J_0 = \gamma\Sigma + (2R + 2r^f - \rho)C \quad (\text{A.45})$$

$$j = Bf - C^\top R^\top A_D(f) - (C^\top R^\top A_{DD} + r^f + R) D^0. \quad (\text{A.46})$$

It follows that

$$x = J^{-1} \left(Bf - (r^f + R)D^0 - C^\top R^\top \hat{V}_D^\top \right) \quad (\text{A.47})$$

$$= J^{-1}j \quad (\text{A.48})$$

and the HJB equation becomes

$$\rho\hat{V} = \frac{1}{2}j^\top J^{-1}j - \left(D^{0\top} A_{DD} + A_D(f)^\top \right) R D^0 + D^{0\top} \mathcal{D}A_D(f) + \mathcal{D}A(f) \quad (\text{A.49})$$

The constant matrix A_{DD} is computed in the usual way:

$$\rho A_{DD} = (A_{DD}RC + r^f + R^\top) J^{-1} (C^\top R^\top A_{DD} + r^f + R) - A_{DD}R - R^\top A_{DD}. \quad (\text{A.50})$$

The coefficient function $A_D(f)$ satisfies the (integro-)differential equation

$$\rho A_D(f) = (A_{DD}RC + r^f + R^\top) J^{-1} (C^\top R^\top A_D(f) - Bf) - R^\top A_D(f) - \mathcal{D}A_D(f), \quad (\text{A.51})$$

and has the representation

$$A_D(f) = - \int_0^\infty \hat{N}_2 e^{-\hat{N}_1 t} B \mathbb{E}[f_t | f_0 = f] \quad (\text{A.52})$$

$$= - \int_0^\infty \hat{N}_2 e^{-\hat{N}_1 t} \hat{N}_3 \mathbb{E}[M_t | f_0 = f], \quad (\text{A.53})$$

with

$$\hat{N}_1 = \rho + R^\top - (A_{DD}RC + r^f + R^\top) J^{-1} \quad (\text{A.54})$$

$$\hat{N}_2 = (A_{DD}RC + r^f + R^\top) J^{-1} C^\top R^\top. \quad (\text{A.55})$$

To prove that the proposed solution does, indeed, solve the trader's problem, we follow a verification argument. Let \hat{x} be an arbitrary trading strategy (satisfying technical transversality conditions) and V quadratic, defined by (22) and the coefficients

A. Since it holds generally that

$$\begin{aligned}
e^{-\rho t}V(\hat{x}_t, \hat{D}_t, f_t) &= e^{-\rho t}\hat{V}(\hat{D}_t - C\hat{x}_t, f_t) - \frac{1}{2}e^{-\rho t}\hat{x}_t^\top C\hat{x}_t \\
&= e^{-\rho T}\hat{V}(\hat{D}_T - C\hat{x}_T, f_T) - \frac{1}{2}e^{-\rho T}\hat{x}_T^\top C\hat{x}_T \\
&\quad - \int_t^T d\left(e^{-\rho s}\hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2}e^{-\rho s}\hat{x}_s^\top C\hat{x}_s\right),
\end{aligned}$$

it would be sufficient that $\lim_{T \rightarrow \infty} e^{-\rho T} E_t \left[\hat{V}(\hat{D}_T - C\hat{x}_T, f_T) - \frac{1}{2}\hat{x}_T^\top C\hat{x}_T \right] = 0$,

$$\begin{aligned}
&E_t \left[- \int_t^T d\left(e^{-\rho s}\hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2}e^{-\rho s}\hat{x}_s^\top C\hat{x}_s\right) \right] \\
&\geq E_t \int_t^T e^{-\rho s} \left(\hat{x}_s^\top (Bf_s - (r+R)\hat{D}_s) - \frac{\gamma}{2}\hat{x}_s^\top \Sigma \hat{x}_s \right) ds \\
&\quad + E_t \int_t^T e^{-\rho s} \hat{x}_s^\top C d\hat{x}_s + \frac{1}{2} E_t \int_t^T e^{-\rho s} d[\hat{x}_s, C\hat{x}_s],
\end{aligned} \tag{A.56}$$

and that the inequality holds with equality at the conjectured optimum control.

Ito's lemma implies

$$\begin{aligned}
&d\left(\hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2}\hat{x}_s^\top C\hat{x}_s\right) \\
&= \hat{V}_D(d\hat{D}_s - Cd\hat{x}_s) + \hat{V}_f(df_s - \Delta f_s) - \hat{x}_s^\top Cd\hat{x}_s - \frac{1}{2}d[\hat{x}_s, C\hat{x}_s] + \\
&\quad \frac{1}{2}d[f_s, \hat{V}_{ff}f_s] - \frac{1}{2}\Delta f_s^\top \hat{V}_{ff}\Delta f_s + \hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \hat{V}(\hat{D}_s - C\hat{x}_s, f_{s-}).
\end{aligned} \tag{A.57}$$

Taking conditional expectations of (A.57), one gets

$$-\hat{V}_D R \hat{D}_s - \hat{x}_s^\top C E_s \frac{1}{ds} [d\hat{x}_s] - \frac{1}{2} \frac{1}{ds} d[\hat{x}_s, C\hat{x}_s] + (\hat{D}_s - C\hat{x}_s)^\top \mathcal{D}A_D(f_s) + \mathcal{D}A(f_s),$$

the negative of which we wish to be larger than

$$\frac{\rho}{2}\hat{x}_s^\top C\hat{x}_s - \rho\hat{V} + \hat{x}_s^\top (Bf_s - (r+R)\hat{D}_s) - \frac{\gamma}{2}\hat{x}_s^\top \Sigma \hat{x}_s + \hat{x}_s^\top C \frac{1}{ds} E_s [d\hat{x}_s] + \frac{1}{2} \frac{1}{ds} E_s d[\hat{x}_s, C\hat{x}_s].$$

This outcome is ensured by the HJB equation (A.43), which is satisfied by \hat{V} and \hat{x} , for any value of \hat{D}^0 . Furthermore, x satisfies the constraint with equality. ■

Proof of Proposition 4. The proof is the same as in ?, and it follows the standard procedure of assuming the functional form (in this case, quadratic) of the value function, computing the optimal control conditional on this function, and then calculating

the coefficients that yield the value function as a fixed point. For completeness, we record the resulting value-function coefficients. With $\bar{\rho} = 1 - \rho\Delta t$ and $\bar{\Lambda} = \bar{\rho}^{-1}\Lambda(\Delta t)$,

$$A_{xx} = \left(\bar{\rho}\gamma\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{\frac{1}{2}} + \frac{1}{4} \left(\rho^2\bar{\Lambda}^2 + 2\rho\gamma\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{\frac{1}{2}} + \gamma^2\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{-1}\Sigma\bar{\Lambda}^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \Delta t \quad (\text{A.58})$$

$$- \frac{1}{2}(\rho\bar{\Lambda} + \gamma\Sigma)\Delta t$$

$$\text{vec}(A_{xf}) = \bar{\rho} \left(I - \bar{\rho}(I - \Phi)^\top \otimes (I - A_{xx}\Lambda(\Delta t)^{-1}) \right)^{-1} \text{vec}((I - A_{xx}\Lambda(\Delta t)^{-1})B). \quad (\text{A.59})$$

■

Proof of Proposition 5. The situation is the same as for Proposition 4. Here, however, we would like to record the equations defining the value-function coefficients, to use in the proof of convergence to the continuous-time solution.

With the additional definitions²

$$\begin{aligned} \Pi &= \begin{bmatrix} \Phi & 0 \\ 0 & R \end{bmatrix} \Delta t, \\ \tilde{C} &= (1 - R\Delta t) \begin{bmatrix} 0 \\ C \end{bmatrix}, \\ \tilde{B} &= [B \quad -(R + r^f)] \Delta t, \end{aligned} \quad (\text{A.60})$$

the unknown matrices have to satisfy the system of equations

$$-\bar{\rho}^{-1}A_{xx} = S_x^\top J^{-1}S_x - \bar{\Lambda} - \bar{\rho}^{-1}C + \tilde{C}^\top A_{yy}\tilde{C} \quad (\text{A.61})$$

$$\bar{\rho}^{-1}A_{xy} = S_x^\top J^{-1}S_y - \tilde{C}^\top A_{yy}(I - \Pi) \quad (\text{A.62})$$

$$\bar{\rho}^{-1}A_{yy} = S_y^\top J^{-1}S_y + (I - \Pi)^\top A_{yy}(I - \Pi), \quad (\text{A.63})$$

where the matrices J , S_x , and S_y are explicit functions of the unknown coefficients:

$$J = \gamma\Sigma\Delta t + \bar{\Lambda} + (2(R + r^f)\Delta t - \bar{\rho}^{-1})C + A_{xx} - 2A_{xy}\tilde{C} - \tilde{C}^\top A_{yy}\tilde{C} \quad (\text{A.64})$$

$$S_x = \bar{\Lambda} + (R + r^f)\Delta t C - A_{xy}\tilde{C} - \tilde{C}^\top A_{yy}\tilde{C} \quad (\text{A.65})$$

$$S_y = \tilde{B} + A_{xy}(I - \Pi) + \tilde{C}^\top A_{yy}(I - \Pi). \quad (\text{A.66})$$

The optimal position x_t can be written as

$$x_t = x_{t-1} + (I - J^{-1}S_x) \left((I - J^{-1}S_x)^{-1} (J^{-1}S_y) y_t \right), \quad (\text{A.67})$$

²We omit notational dependence on Δt for simplicity.

so that

$$M^{rate} = I - J^{-1}S_x \tag{A.68}$$

$$M^{aim} = (I - J^{-1}S_x)^{-1} (J^{-1}S_y). \tag{A.69}$$

■

Proof of Proposition 6. In the text. ■

Proof of Proposition 7. In the text. ■

Proof of Proposition 8. Consider evaluating the objective (34) at $x_t^{(\Delta t)}$ and letting Δt go to zero. The only non-standard terms in the objective are the ones involving transaction costs:

$$E_0 \left[\sum_t - (1 - \rho\Delta t)^{t+1} x_t^\top (R + r^f) (D_t + C\Delta x_t) \Delta t \right. \\ \left. + (1 - \rho\Delta t)^t \left(-\frac{1}{2} \Delta x_t^\top \Lambda(\Delta t) \Delta x_t + x_{t-1}^\top C \Delta x_t + \frac{1}{2} \Delta x_t^\top C \Delta x_t \right) \right]. \tag{A.70}$$

Under regularity conditions, the sum of the terms not involving Λ tends to

$$E_0 \int_t^\infty -e^{-\rho t} (x_t^\top (r^f + R) D_t) dt \tag{A.71} \\ + E_0 \int_0^\infty e^{-\rho t} x_{t-}^\top C dx_t + \frac{1}{2} E_0 \int_0^\infty e^{-\rho t} d[x, Cx]_t.$$

If we write $\Lambda(\Delta t) = \Lambda s(\Delta t)$ for some scalar function s and let $\tau_t = \frac{1}{dt} E_t[dx_t]$, then we can express the remaining limit term as

$$-\frac{1}{2} E_0 \int_0^\infty e^{-\rho t} \tau_t^\top \Lambda \tau_t dt \times \lim_{\Delta t \rightarrow 0} (s(\Delta t) \Delta t) - \frac{1}{2} E_0 \int_0^\infty e^{-\rho t} d[x, \Lambda x]_t \times \lim_{\Delta t \rightarrow 0} s(\Delta t). \tag{A.72}$$

We note that, under the assumption of part (i), i.e., $s(\Delta t) = \Delta t^{-1}$, the first term is non-zero while the second is infinite if x has non-zero quadratic variation. Under the assumption of part (ii), i.e., $s(\Delta t) = \Delta t$, both terms are zero, so that the objective coincides with (21).

The only element that may need proving, therefore, is that the optimal trade in the discrete-time model has a well-defined continuous-time limit; for if the limit exists, it has to be optimal, or else its discretely sampled counterpart can be improved upon,

at least for Δt small enough. We know that the discrete-time optimal trade is given as a quadratic function of the exogenous process $\{f_t\}_t$, so the only claim to prove is that the sequence of matrix tuples (A_{xx}, A_{xy}, A_{yy}) , which generate the coefficients of this function, has a (finite) limit.

We can achieve this goal through direct manipulation of the Riccati equations defining the discrete-time coefficients and taking the limit. The precise details depend on the case of the proposition and the coefficient, but the general idea is the same. We illustrate for the matrix A_{xx} under the assumption $\Lambda(\Delta t) = \Delta t^{-1}\Lambda$.

We work with the characterization of solutions provided in the proof of Proposition 5. We show that, as $\Delta t \rightarrow 0$, Equation (A.61) tends to its counterpart in (A.35), which implies that the solutions also do.

We first rewrite this equation as

$$-A_{xx} = \bar{\rho} S_x^\top J^{-1} S_x - \Lambda(\Delta t) - C + \bar{\rho} \tilde{C}^\top A_{yy} \tilde{C} \quad (\text{A.73})$$

$$= \bar{\rho} (S_x - J)^\top J^{-1} (S_x - J) - \Lambda(\Delta t) - \bar{\rho} J + 2\bar{\rho} S_x + \bar{\rho} \tilde{C}^\top A_{yy} \tilde{C}, \quad (\text{A.74})$$

and then rearrange it, using (A.64) and (A.65), as

$$-A_{xx} (1 - \bar{\rho}) = \bar{\rho} (S_x - J)^\top J^{-1} (S_x - J) - \bar{\rho} \gamma \Sigma \Delta t. \quad (\text{A.75})$$

Dividing through by Δt and ignoring terms in Δt in $S_x - J$ and $J\Delta t$ — note that $\bar{\Lambda}(\Delta t)\Delta t \rightarrow \Lambda$ as $\Delta t \rightarrow 0$ — we obtain

$$-\rho A_{xx} = -\gamma \Sigma + \left(A_{xx} - A_{xy} \tilde{C} - C \right) \Lambda^{-1} \left(A_{xx} - \tilde{C}^\top A_{xy}^\top - C^\top \right), \quad (\text{A.76})$$

the same as in continuous time.

Once the value-function coefficients in discrete time are established to have as limit their counterparts in continuous time, one proceeds by noting that, when letting Δt go to 0, the rate term M^{rate} is given by

$$\Lambda^{-1} \left(A_{xx} - C - \tilde{C} A_{xy}^\top \right), \quad (\text{A.77})$$

while the aim term M^{aim} by

$$\left(A_{xx} - C^\top A_{xy}^\top - C^\top \right)^{-1} \left(A_{xy} + \tilde{C}^\top A_{yy} \right). \quad (\text{A.78})$$

These expressions are the same as obtained in continuous time. ■

Proof of Proposition 9. We start from the HJB equation, which reads

$$0 = \sup_{\tau_s} \left\{ x_s^\top B f_s - \frac{\gamma}{2} x_s^\top \bar{\Sigma} x_s v_s - \frac{\lambda}{2} \tau_s^\top \bar{\Sigma} \tau_s v_s - \rho V + V_x \tau_s - V_f \Phi f + \frac{1}{2} \text{tr}(V_{ff} \Omega) \right. \\ \left. + V_v \mu_v + \frac{1}{2} V_{vv} \sigma_v^2 + V_{vf} \frac{d}{ds} [f, v]_s \right\}, \quad (\text{A.79})$$

with

$$V_v = -\frac{1}{2} x^\top A'_{xx} x + x^\top A'_{xf} f + \frac{1}{2} f^\top A'_{ff} f + A'_0 \quad (\text{A.80})$$

$$V_{vv} = -\frac{1}{2} x^\top A''_{xx} x + x^\top A''_{xf} f + \frac{1}{2} f^\top A''_{ff} f + A''_0. \quad (\text{A.81})$$

Similarly to the special case of Proposition 1, we conjecture and verify that $A_{xx}(v) = a(v) \bar{\Sigma}$. Under the assumption, collecting the terms in (A.79) that are quadratic in x gives rise to the ODE

$$0 = \lambda^{-1} \frac{a^2}{v^2} + \rho \frac{a}{v} - \gamma - \frac{a'}{v} \mu_v - \frac{1}{2} \frac{a''}{v} \sigma_v^2. \quad (\text{A.82})$$

The first observation we make is that a is increasing, since the value function is unambiguously decreasing in v . Let a_0 be the constant solving $\lambda^{-1} a^2 + \rho a = \gamma$ (i.e., given by (A.18)) and v_z the point where $\mu_v(v_z) = 0$.

Suppose now that, for some value v , $a(v) = a_0 v$. If $\mu_v(v) < 0$, then (A.82) implies that $a''(v) > 0$. If, furthermore, $a'(v) \geq a_0$, then $a(v') > a_0 v'$ for all $v' > v$. Thus, if a crosses $a_0 v$ from below once $\mu_v < 0$, then it remains above $a_0(v)$ for all v . On the other hand, for v sufficiently high, $a(v) < a_0 v$. This statement holds because $a(v)$ is bounded above by the utility generated by any suboptimal policy. In particular, consider a policy with

$$M^{\text{rate}}(v) = \lambda^{-1} a_0, \quad (\text{A.83})$$

and let \hat{a} be the resulting value-function coefficient. Then \hat{a} solves

$$0 = -\frac{a_0^2}{\lambda} + \rho \frac{\hat{a}}{v} - \gamma + 2 \frac{a_0 \hat{a}}{\lambda v} - \frac{\hat{a}'}{v} \mu_v - \frac{1}{2} \frac{\hat{a}''}{v} \sigma_v^2, \quad (\text{A.84})$$

or

$$\hat{a}(v) = \int_0^\infty e^{-(\rho+2\lambda^{-1}a_0)t} (\lambda^{-1} a_0^2 + \gamma) \text{E}_0[v_t | v_0 = v] dt. \quad (\text{A.85})$$

It follows that, for $v > v_z$,

$$\hat{a}(v) > \frac{\lambda^{-1}a_0^2 + \gamma}{\rho + 2\lambda^{-1}a_0}v \quad (\text{A.86})$$

$$= a_0v. \quad (\text{A.87})$$

In conclusion, for $v > v_z$, a cannot cross the line a_0v from below.

Consider now a point $v < v_z$ where a crosses a_0v from above. Here it must be the case that $a''(v) < 0$, so that $a'(v) < a_0$ for all $v' \in (v, v_z)$. Thus a cannot cross a_0v on $(0, v_z)$ again after crossing downwards the first time. Finally, it is obvious that $a(0) > 0$. (If v is bounded below away from zero a.s., then, as long as σ_v is zero at the lower bound \underline{v} , (A.82) ensures that $a(\underline{v}) > a_0\underline{v}$.)

In conclusion, $a(v)$ starts above a_0v and ends below it, and can never cross it upwards. The unique crossing point is the desired \hat{v} . The conclusion of the proposition follows immediately from the fact that the trading rate with constant v is $\lambda^{-1}a_0$. ■

Proof of Proposition 10. Note first that the aggregate noise-trader holding z_t satisfies

$$dz_t = \kappa \left(\sum_{l=1}^L f_t^l - z_t \right) dt. \quad (\text{A.88})$$

Given the definition of f , the mean-reversion matrix Φ is given by

$$\Phi = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa & -\kappa & \cdots & \kappa \end{pmatrix}. \quad (\text{A.89})$$

Supposing that $E_t[dp_t - r^f p_t dt] = B f_t dt$, we use the results in Proposition 1 together with the market-clearing conditions to derive

$$\frac{a}{\lambda} \sigma^{-2} B (a\Phi + \gamma I)^{-1} + \frac{a}{\lambda} e_{L+1} = -\kappa(\mathbf{1} - 2e_{L+1}), \quad (\text{A.90})$$

where $e_{L+1} = (0, \dots, 0, 1) \in \mathbb{R}^{L+1}$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{L+1}$. It consequently follows that, if the investor is to hold $-z_t = -f_t^{L+1}$ at time t for all t , then the factor loadings must be given by

$$B = \sigma^2 \left[-\frac{\lambda}{a} \kappa(\mathbf{1} - 2e_{L+1}) - e_{L+1} \right] (a\Phi + \gamma I). \quad (\text{A.91})$$

For $l \leq L$, we calculate B_l further as

$$\begin{aligned} B_l &= -\sigma^2 \kappa (\lambda \psi_l + \lambda \gamma a^{-1} + \lambda \kappa - a) \\ &= -\lambda \sigma^2 \kappa (\psi_l + \rho + \kappa), \end{aligned} \tag{A.92}$$

while

$$B_{L+1} = \sigma^2 (\rho \lambda \kappa + \lambda \kappa^2 - \gamma). \tag{A.93}$$

We have thus shown that a (unique) matrix B exists such that, if $E_t[dp_t - r^f p_t dt] = B f_t dt$, then the market is in equilibrium. The comparative-static results are immediate. ■